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Exercises in exact quantization

André Voros

CEA, Service de Physique Théorique de Saclay, F-91191 Gif-sur-Yvette Cedex, France
and
Institut de Mathématiques de Jussieu–Chevaleret, CNRS UMR 7586, Université Paris 7, 2 place Jussieu, F-75251 Paris Cedex 05, France

E-mail: voros@spht.saclay.cea.fr

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Abstract. The formalism of exact 1D quantization is reviewed in detail and applied to the spectral study of three concrete Schrödinger Hamiltonians $[-\mathrm{d}^2/\mathrm{d}q^2 + V(q)]^{\pm}$ on the half-line $\{q > 0\}$, with a Dirichlet (−) or Neumann (+) condition at $q = 0$. Emphasis is put on the analytical investigation of the spectral determinants and spectral zeta-functions with respect to singular perturbation parameters. We first discuss the homogeneous potential $V(q) = q^N$ as $N \to +\infty$ versus its (solvable) $N = \infty$ limit (an infinite square well): useful distinctions are established between regular and singular behaviours of spectral quantities; various identities among the square-well spectral functions are unravelled as limits of finite-N properties. The second model is the quartic anharmonic oscillator: the zero-energy spectral determinants $\det(-\mathrm{d}^2/\mathrm{d}q^2 + q^4 + vq^2)^{\pm}$ are explicitly analysed in detail, revealing many special values, algebraic identities between Taylor coefficients and functional equations of a quartic type coupled to asymptotic $v \to \infty$ properties of Airy type. The third study addresses the potentials $V(q) = q^N + vq^{N/2-1}$ of even degree: their zero-energy spectral determinants prove computable in closed form, and the generalized eigenvalue problems with v as spectral variable admit exact quantization formulae which are perfect extensions of the harmonic oscillator case (corresponding to $N = 2$); these results partly reflect the presence of quasi-exactly solvable potentials in the family above.

Exact quantization, or exact WKB analysis, supplies new tools for the analytical study of the 1D Schrödinger equation, now including arbitrary polynomial potentials. Here we initiate applications of such an exact method to miscellaneous concrete problems and models of analytical interest, emphasizing exact and asymptotic relations for the spectral determinants and related spectral zeta-functions.

We have chosen three rather different quantum potentials to illustrate a variety of situations. These have some basic common features (besides their required 1D and polynomial nature): they are rather simple, with a minimal number of parameters, to remain concretely manageable; one crucial parameter (discrete or continuous) governs the transition to a singular limit, creating an interesting dynamical and analytical situation; some uniform principles for tackling those problems can be issued at the most general level.

In contrast to earlier studies concerned with individual eigenvalue or eigenfunction behaviour, we seek the limiting properties of spectral functions, which are symmetric functions of all the eigenvalues at once. We take a semi-rigorous approach, in which we argue a global operational scheme without claiming absolute completeness in every detail.

The paper is organized as follows.

The introduction, section 1, gives a detailed survey of the exact tools to be used here, while an appendix collects all the results concerning homogeneous potentials. This is done partly for convenience, given the lack of comprehensive reviews for results that are very scattered in

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time and place of publication, and also to clarify some parts of our most recent developments where localized inconsistencies went undetected.

Then, section 2 (exercise 1) considers the family of homogeneous potentials $V(q) = |q|^N$ as the degree N tends to $+\infty$. This is a most singular problem, for which many explicit results are however available beforehand, and the limiting problem (an infinite square well) is exactly solved by elementary means. We therefore mainly propose and test some general principles of investigation, rather than claim truly new results. In particular, we suggest criteria for sorting out regular versus singular types of limiting behaviour in spectral zeta-functions and determinants. Still, we identify several possibly unnoticed properties and formulae in the $N = \infty$ limit which arise as regular limits of nontrivial finite-N properties.

Section 3 (exercise 2) deals with the quartic anharmonic oscillator family $V(q) = q^4+vq^2$, which is the most common model for singular perturbation theory (the free harmonic oscillator emerges in the $v \to +\infty$ limit). We single out a pair of one-parameter spectral functions for their remarkably numerous and simple explicit properties: the zero-energy determinants $\mathrm{Qi}^{\pm}(v) \stackrel{\mathrm{def}}{=} \det(-\mathrm{d}^2/\mathrm{d}q^2+q^4+vq^2)^{\pm}$ (+ corresponds to the even-state sector and − to the odd-state sector). We present a simple WKB technique allowing us to express asymptotic relations between v-dependent determinants such as $\det(-\mathrm{d}^2/\mathrm{d}q^2+q^4+vq^2)^{\pm}$ and $\det(-\mathrm{d}^2/\mathrm{d}q^2+vq^2)^{\pm}$ when $v \to \infty$. Then, practically all the analytical results available for the homogeneous quartic case ($V(q) = q^4$) have counterparts for the functions $\mathrm{Qi}^{\pm}$ (and their associated spectral zeta functions), while the plots and asymptotic properties of $\mathrm{Qi}^{\pm}$ evoke the Airy functions. Several special values are computable and are tabulated against the analogous results for the Airy functions and the quartic determinants $\det(-\mathrm{d}^2/\mathrm{d}q^2 + q^4 + \lambda)^{\pm}$.

The final section 4 (exercise 3) gives a complete treatment of the similar determinants for a different class of binomial potentials, $V(q) = q^N + vq^{N/2-1}$ (for N even). Here the formalism yields fully closed forms for the zero-energy spectral determinants: equation (120) for $\det[-\mathrm{d}^2/\mathrm{d}q^2 + V(|q|)]^{\pm}$ in terms of gamma functions (plus nontrivial exponential prefactors) and, when N is a multiple of 4, the still simpler equation (123) for the zero-energy determinant of the potential $V(q)$ itself on the whole real line; exact quantization formulae follow for the corresponding generalized spectra (in the v variable: equations (107) and (124) respectively). A broad generalization of the familiar harmonic-oscillator exact results is thus obtained; for $N \equiv 2$ [mod 4], this seems to describe a zero-energy cross-section of the formalism for *quasi-exactly solvable* models.

Although all three problems rely on the same background formalism, they can be approached fairly independently from one another. Accordingly, there are no global conclusions but each section carries its own concluding remarks.

1. Introduction

1.1. General results on spectral functions [1, 2]

Here, an *admissible* spectrum is a purely discrete countable set $\{E_k\}_{k=0,1,2,\ldots}$ with $E_k > 0$, $E_k \uparrow +\infty$, such that its partition function

$$\theta(t) \stackrel{\mathrm{def}}{=} \sum_{k=0}^{\infty} \mathrm{e}^{-tE_k} \qquad \mathrm{Re}\,(t) > 0 \tag{1}$$

can be asymptotically expanded in increasing (real) powers $\{t^\rho\}$ for $t \downarrow 0$:

$$\theta(t) \sim \sum_\rho c_\rho t^\rho \ (t \downarrow 0) \qquad \text{with} \quad \mu \stackrel{\mathrm{def}}{=} -\min\{\rho\} > 0 \text{ (the 'growth order')}. \tag{2}$$

(In summations etc Latin indices will systematically mean integers, and Greek indices mean more general real indices, namely real-valued functions over the natural integers, strictly $\uparrow +\infty$ or $\downarrow -\infty$.)

1.1.1. Spectral zeta-functions [2–4]. Equation (2) implies that the 'Hurwitz', respectively 'plain' spectral zeta-functions

$$Z(s,\lambda) \stackrel{\text{def}}{=} \sum_k (E_k+\lambda)^{-s} \quad (|\arg(\lambda+E_0)| < \pi-\delta) \qquad \text{respectively } Z(s) \stackrel{\text{def}}{=} Z(s,0) \tag{3}$$

converge for $\mathrm{Re}\,(s) > \mu$, that $Z(s)$ has a meromorphic continuation to all complex s, with [5]

$$\text{polar set:} \quad \{-\rho\} \qquad \text{residue formula:} \quad \lim_{s\to-\rho}(s+\rho)Z(s) = c_\rho/\Gamma(-\rho) \tag{4}$$

$$\text{'trace identities':} \quad \text{for } m \in \mathbb{N} \qquad Z(-m) = (-1)^m m!\, c_m \qquad (c_m \stackrel{\text{def}}{=} 0 \text{ for } m \notin \{\rho\}) \tag{5}$$

and similarly for $Z(s,\lambda)$, just by substituting $\mathrm{e}^{-\lambda t}\theta(t)$ for $\theta(t)$; for general λ, the leading trace identity (especially useful for equation (15) below) is then polynomial, being

$$Z(0,\lambda) = \sum_{0\leqslant n\leqslant\mu} c_{-n}\frac{(-\lambda)^n}{n!} \equiv Z(0) + \sum_{1\leqslant n\leqslant\mu} \mathrm{Res}_{s=n} Z(s)\frac{(-\lambda)^n}{n}. \tag{6}$$

1.1.2. Spectral determinants [1]. Since $Z(s,\lambda)$ is regular at $s=0$, a spectral determinant can be defined by zeta-regularization, as

$$D(\lambda) \equiv \det(\hat{H}+\lambda) \stackrel{\text{def}}{=} \exp[-\partial_s Z(s,\lambda)]_{s=0} \tag{7}$$

($\hat{H}$ being a linear operator of spectrum $\{E_k\}$) and it is an entire function of order μ with $\{-E_k\}$ as its set of zeros. Moreover, amidst all such functions, $D(\lambda)$ can be precisely picked out in at least two ways.

- On one hand, equation (2) implies a canonical semiclassical behaviour for $D(\lambda)$:

$$-\log D(\lambda) \sim \sum_\rho c_\rho \Gamma_\rho(\lambda) \qquad \text{for} \quad \lambda \to +\infty \tag{8a}$$

  with

$$\Gamma_\rho(\lambda) \stackrel{\text{def}}{=} \partial_s\left[\frac{\Gamma(s+\rho)}{\Gamma(s)}\lambda^{-(s+\rho)}\right]_{s=0} \tag{8b}$$

  that is,

$$\Gamma_\rho(\lambda) = \begin{cases} \Gamma(\rho)\lambda^{-\rho} & \text{if} \quad -\rho \notin \mathbb{N} \\ (-(-\lambda)^m/m!)\left(\log\lambda - \sum_{r=1}^{m} 1/r\right) & \text{if} \quad -\rho = m \in \mathbb{N} \end{cases} \tag{8c}$$

  are the *only* terms allowed; no other type, including additive constants ($\propto \lambda^0$), can enter this expansion.
- Independently, $D(\lambda)$ is also fully specified by expansions around $\lambda = 0$: firstly, in reference to the Fredholm determinant $\Delta(\lambda)$ (built as a Weierstrass infinite product),

$$D(\lambda) \equiv \exp\left[-Z'(0) - \sum_{1\leqslant n\leqslant\mu}\frac{\tilde{Z}(n)}{n}(-\lambda)^n\right]\Delta(\lambda) \tag{9}$$

  where

$$\Delta(\lambda) \stackrel{\text{def}}{=} \prod_{k=0}^{\infty}\left(1+\frac{\lambda}{E_k}\right)\exp\left[\sum_{1\leqslant n\leqslant\mu}\frac{(-\lambda)^n}{nE_k^n}\right] \qquad \text{for all } \lambda \tag{10}$$

and

$$\tilde{Z}(n) = Z(n) \qquad \text{if} \quad Z(s) \text{ is regular at } n \text{ (as when } n > \mu) \tag{11}$$

$$\tilde{Z}(1) = \lim_{s\to 1}\left(Z(s) - \frac{c_{-1}}{s-1}\right) \qquad (=Z(1) \text{ or its finite part}) \tag{12}$$

(the general formula for $\tilde{Z}(n)$ on a pole is more contrived and required only when $\mu \geqslant 2$ [1, equation (4.12)], whereas $\mu \leqslant \frac{3}{2}$ in this work). Finally, by way of consequence, the determinants are also characterized by these Taylor series (converging for $|\lambda| < E_0$),

$$\begin{aligned} -\log D(\lambda) &= Z'(0) + \sum_{n=1}^{\infty} \frac{\tilde{Z}(n)}{n}(-\lambda)^n \\ -\log \Delta(\lambda) &= \sum_{n>\mu} \frac{Z(n)}{n}(-\lambda)^n. \end{aligned} \tag{13}$$

The simplest case is $\mu < 1$: then

$$\Delta(\lambda) = \prod_{k=0}^{\infty}\left(1 + \frac{\lambda}{E_k}\right) = \exp\left[-\sum_{n=1}^{\infty}\frac{Z(n)}{n}(-\lambda)^n\right] \qquad D(\lambda) \equiv \mathrm{e}^{-Z'(0)}\Delta(\lambda). \tag{14}$$

Two other properties are worth mentioning:

- if a spectrum $\{E_k\}$ is dilated to $\{\alpha E_k\}$ ($\alpha > 0$), the spectral functions are rescaled to

$$Z(s, \lambda|\alpha) \equiv \alpha^{-s} Z(s, \lambda/\alpha) \Longrightarrow D(\lambda|\alpha) \equiv \alpha^{Z(0,\lambda/\alpha)} D(\lambda/\alpha) \tag{15}$$

  (a behaviour hence mainly governed by the leading trace identity, equation (6));
- finally, all these results extend to analogous complex spectra [6].

1.2. 1D Schrödinger operators with polynomial potentials

We subsequently specialize to 1D Schrödinger equations involving a polynomial potential $V(q)$ (adjusted to $V(0) = 0$) [7, 8]

$$(-\mathrm{d}^2/\mathrm{d}q^2 + [V(q) + \lambda])\psi = 0 \qquad V(q) = +q^N + [\text{lower-order terms}]. \tag{16}$$

We call $\hat{H}^+$ (respectively $\hat{H}^-$) the Schrödinger operator on the half-line $\{q > 0\}$ with the Neumann (respectively Dirichlet) boundary condition at $q = 0$, and $\hat{H}$ the Schrödinger operator on the whole line with the potential $V(|q|)$, whose spectrum we denote by $\{E_k\}$. Then $\{E_k\}_{k \text{ even}}$ (respectively $\{E_k\}_{k \text{ odd}}$) is the spectrum of $\hat{H}^+$ (respectively $\hat{H}^-$); each one is an admissible spectrum in the previous sense, with

$$\text{growth order } \mu = \frac{1}{2} + \frac{1}{N} \qquad (\mu < 1 \text{ generically, i.e. for } N > 2) \tag{17}$$

$$\text{exponents } \{\rho\} = \{-\mu + j/N\}_{j=0,1,2,\ldots}. \tag{18}$$

Their spectral functions Z^+, D^+ (respectively Z^-, D^-) are our basic concern, and have properties as above (with exceptions in the singular case $N = 2$). However, a few results take a neater or a more regular form upon recombined functions instead,

$$Z \stackrel{\text{def}}{=} Z^+ + Z^- \qquad D \stackrel{\text{def}}{=} D^+ D^- \qquad (\text{spectral functions of } \hat{H}) \tag{19}$$

$$Z^{\mathrm{P}} \stackrel{\text{def}}{=} Z^+ - Z^- \qquad D^{\mathrm{P}} \stackrel{\text{def}}{=} D^+/D^- \qquad (\text{‘skew’ spectral functions}). \tag{20}$$

1.2.1. Classical 'spectral' functions [9]. The quantum spectral functions of the problem (16) admit natural classical counterparts with parallel properties. The Weyl–Wigner correspondence, for instance, associates the following classical partition function with the quantum one of equation (1):

$$\theta_{\mathrm{cl}}(t) = \int_{\mathbb{R}^2} \frac{\mathrm{d}p\,\mathrm{d}q}{2\pi} \mathrm{e}^{-[p^2+V(|q|)]t} \equiv \frac{1}{\sqrt{\pi t}} \int_0^{+\infty} \mathrm{e}^{-V(q)t}\,\mathrm{d}q \tag{21}$$

with an expansion coinciding with equation (2) as long as $\rho \leqslant 0$. The same Mellin transforms as from the quantum $\theta(t)$ to $Z(s,\lambda)$ then yield

$$Z_{\mathrm{cl}}(s,\lambda) \stackrel{\mathrm{def}}{=} \frac{1}{\Gamma(s)} \int_0^{+\infty} \theta_{\mathrm{cl}}(t)\mathrm{e}^{-\lambda t} t^{s-1}\,\mathrm{d}t = \frac{\Gamma(s-\frac{1}{2})}{\Gamma(s)\sqrt{\pi}} I_0(s,\lambda) \tag{22}$$

where

$$I_q(s,\lambda) \stackrel{\mathrm{def}}{=} \int_q^{+\infty} (V(q')+\lambda)^{-s+1/2}\,\mathrm{d}q' \qquad (\mathrm{Re}\,(s) > \mu) \tag{23}$$

and

$$D_{\mathrm{cl}}(\lambda) \stackrel{\mathrm{def}}{=} \exp[-\partial_s Z_{\mathrm{cl}}(s,\lambda)]_{s=0} \tag{24}$$

with properties induced by equation (2) similar to the quantum case (but $D_{\mathrm{cl}}(\lambda)$ is not an entire function: branch cuts replace the chains of discrete zeros of $D(\lambda)$).

The meromorphic continuation of $I_q(s,\lambda)$ is thus important at $s = 0$. If (for $\lambda > -\inf V$ initially) we compute the expansion

$$(V(q)+\lambda)^{-s+1/2} \sim \sum_\sigma \beta_\sigma(s) q^{\sigma - Ns} \qquad \text{for} \quad q \to +\infty \qquad \left(\sigma = \frac{N}{2}, \frac{N}{2}-1, \ldots\right) \tag{25}$$

(the β_σ of course also depend on λ and on the potential), then

$$I_q(s,\lambda) \sim -\sum_\sigma \beta_\sigma(s) \frac{q^{\sigma+1-Ns}}{\sigma+1-Ns} \qquad (q \to +\infty) \tag{26}$$

hence it satisfies

$$\lim_{s\to 0} s I_q(s,\lambda) = \beta_{-1}(0)/N = -Z(0,\lambda)/2. \tag{27}$$

$\beta_{-1}(s)$ is actually independent of λ except for $N = 2$; the latter value for the residue (27) comes directly from equation (22), making another explicit statement of the trace identity (6).

We denote here

$$\mathcal{I}_q(\lambda) \stackrel{\mathrm{def}}{=} \lim_{s\to 0} \left\{ I_q(s,\lambda) - \frac{\beta_{-1}(0)}{Ns} \right\} \qquad \text{(the finite part of } I_q(s,\lambda) \text{ at } s=0). \tag{28}$$

Now, some of our earlier statements revolving around this quantity [9, 10] require corrections in the most general setting $\beta_{-1}(s) \not\equiv 0$. Finite parts are also to be extracted: first on the expansion (26), giving

$$\mathcal{I}_q(\lambda) \sim -\sum_{\sigma\neq -1} \beta_\sigma(0) \frac{q^{\sigma+1}}{\sigma+1} - \beta_{-1}(0)\log q + \frac{\partial_s \beta_{-1}(0)}{N} \qquad (q \to +\infty) \tag{29}$$

then on the definition (22), (24), giving

$$\frac{1}{2}\log D_{\mathrm{cl}}(\lambda) \stackrel{\mathrm{def}}{=} -\frac{1}{2}\partial_s Z_{\mathrm{cl}}(s,\lambda)_{s=0} = \mathcal{I}_0(\lambda) + 2(1-\log 2)\frac{\beta_{-1}(0)}{N}. \tag{30}$$

Next, the finite part $\mathcal{I}_0(\lambda)$ of $\int_0^{+\infty}(V(q)+\lambda)^{-s+1/2}\,\mathrm{d}q$ at $s=0$ is a candidate to define a 'symbolic' value for the divergent integral $\int_0^{+\infty}\Pi_\lambda(q)\,\mathrm{d}q$, where

$$\Pi_\lambda(q) \stackrel{\text{def}}{=} (V(q)+\lambda)^{1/2} \qquad \text{the classical (forbidden-region) momentum.} \tag{31}$$

However, such an assignment being conventional, we much prefer the 'renormalization' given by equation (30), which only adds an explicit constant to the finite part. So (more generally) we pose the suggestive notation

$$\int_q^{+\infty}\Pi_\lambda(q')\,\mathrm{d}q' \stackrel{\text{def}}{=} \frac{1}{2}\log D_{\mathrm{cl}}(\lambda) - \int_0^q \Pi_\lambda(q')\,\mathrm{d}q' \equiv \mathcal{I}_q(\lambda) + 2(1-\log 2)\frac{\beta_{-1}(0)}{N}. \tag{32}$$

A big advantage of the specification (32) will be that its $\lambda \to +\infty$ expansion has the canonical form (8) (essentially because $\log D_{\mathrm{cl}}$ behaves similarly to its quantum counterpart $\log D$ in this respect, and $\int_0^q \Pi_\lambda(q')\,\mathrm{d}q' = \mathrm{O}(\sqrt{\lambda}) + \mathrm{o}(1)$ is also manifestly canonical).

Thanks to equation (29), the prescription (32) is also directly characterized by its large-q asymptotic behaviour, as

$$\int_q^{+\infty}\Pi_\lambda(q')\,\mathrm{d}q' = -\mathcal{S}_\lambda(q) - \beta_{-1}(0)\log q + \mathcal{C} + \mathrm{o}(1) \qquad (q \to +\infty) \tag{33}$$

where

$$\mathcal{S}_\lambda(q) \stackrel{\text{def}}{=} \sum_{\{\sigma>-1\}} \beta_\sigma(0)\frac{q^{\sigma+1}}{\sigma+1} \qquad \mathcal{C} \stackrel{\text{def}}{=} \frac{1}{N}\left(-2\log 2\,\beta_{-1}(0) + \partial_s\left[\frac{\beta_{-1}(s)}{1-2s}\right]_{s=0}\right).$$

Classical analogues will also arise for D^+ and D^- separately (equation (46) below).

1.2.2. Special features of 1D Schrödinger determinants [9, 10]. The quantum determinants $D^\pm$ for equation (16) can be specified in two additional ways.

- The spectrum of either $\hat{H}^+$ or $\hat{H}^-$ obeys a semiclassical (high-energy Bohr–Sommerfeld) quantization condition of the form

$$\sum_\rho b^\pm_{-\rho} E_k^{-\rho} \sim k + \tfrac{1}{2} \qquad \text{for } {}^{\text{even}}_{\text{odd}}\ k \to +\infty \tag{34}$$

which implies Euler–Maclaurin continuation formulae for the zeta-functions; e.g. down to $\mathrm{Re}\,(s) \geqslant 0$,

$$Z^\pm(s) = \lim_{K\to+\infty}\left\{\sum_{k<K} E_k^{-s} + \frac{1}{2}E_K^{-s} - \frac{1}{2}\sum_{\{\rho<0\}}\frac{\rho\, b^\pm_{-\rho}}{s+\rho}E_K^{-s-\rho}\right\} \qquad \text{for } {}^{\text{even}}_{\text{odd}}\ k, K. \tag{35}$$

The resulting residues are compatible with equation (4) provided

$$\frac{b^\pm_{-\rho}}{2} \equiv \frac{c^\pm_\rho}{\Gamma(1-\rho)} \quad (\rho \neq 0) \qquad \frac{b^\pm_0}{2} \pm \frac{1}{4} \equiv c^\pm_0 \equiv Z^\pm(0) \tag{36}$$

(the latter specifies the $s=0$ trace identities); here, moreover,

$$b^+_{-\rho} \equiv b^-_{-\rho} \quad \text{for all } \rho \leqslant 0 \qquad \Longrightarrow \qquad Z(0) = b^\pm_0 \qquad Z^{\mathrm{P}}(0) = \tfrac{1}{2} \tag{37}$$

then the same trace identity follows for all $Z(0,\lambda)$ and $Z^{\mathrm{P}}(0,\lambda)$ by equation (6), except in the case $N=2$ (equation (71) below).

By the same token, $D^{\pm}(\lambda)$ become directly specifiable as functionals of the spectrum: in the simplest case $\mu < 1$, i.e. $N > 2$,

$$\log D^{\pm}(\lambda) = \lim_{K\to+\infty} \Bigg\{ \sum_{k<K} \log(E_k+\lambda) + \tfrac{1}{2}\log(E_K+\lambda) - \frac{1}{2}\sum_{\{\rho<0\}} b^{\pm}_{-\rho} E_K^{-\rho}\left(\log E_K + \frac{1}{\rho}\right)\Bigg\} \qquad \text{for } k, K \ {}^{\text{even}}_{\text{odd}} \tag{38}$$

(the expansion coefficients $b^{\pm}_{-\rho}$ being themselves functions of the spectrum $\{E_k\}$).

- $D^{\pm}(\lambda)$ are also related to an exact solution $\psi_\lambda(q)$ of equation (16) defined by a particular WKB normalization,

$$\psi_\lambda(q) \sim \Pi_\lambda(q)^{-1/2} \mathrm{e}^{\int_q^{+\infty}\Pi_\lambda(q')\,\mathrm{d}q'} \qquad \text{for} \quad \Pi_\lambda(q) \to +\infty \tag{39}$$

where $\Pi_\lambda(q)$ is the classical momentum function (31), and the divergent integral $\int_q^{+\infty}\Pi_\lambda(q')\,\mathrm{d}q'$ is specifically 'renormalized' through equation (32); however, equation (33) also makes $\psi_\lambda(q)$ directly specified as the (unique) solution of the Schrödinger equation (16) that decays for $q \to +\infty$ with the precise behaviour

$$\psi_\lambda(q) \sim \mathrm{e}^{\mathcal{C}} q^{-N/4-\beta_{-1}(0)} \mathrm{e}^{-\mathcal{S}_\lambda(q)} (\equiv \mathrm{e}^{\mathcal{C}} q^{NZ^-(0,\lambda)} \mathrm{e}^{-\mathcal{S}_\lambda(q)}) \qquad q \to +\infty \tag{40}$$

(the latter form comes from equations (27) and (37)). We remark that, in parallel to equations (39) and (40),

$$-\psi'_\lambda(q) \sim \Pi_\lambda(q)^{+1/2} \mathrm{e}^{\int_q^{+\infty}\Pi_\lambda(q')\,\mathrm{d}q'} \qquad \text{for} \quad \Pi_\lambda(q) \to +\infty \tag{41}$$

$$(\sim \mathrm{e}^{\mathcal{C}} q^{+N/4-\beta_{-1}(0)} \mathrm{e}^{-\mathcal{S}_\lambda(q)} = \mathrm{e}^{\mathcal{C}} q^{NZ^+(0,\lambda)} \mathrm{e}^{-\mathcal{S}_\lambda(q)} \qquad \text{for } q \to +\infty).$$

We refer to $\psi_\lambda(q)$ as the 'canonical recessive' solution. (It is proportional to the 'subdominant' solution of [7, Chapter 2], but only equal to it if $\beta_{-1}(s) \equiv 0$. The current normalization also differs from [11], and from [9, 10]—where it suffers localized inconsistencies. The discrepancies, which for $N > 2$ involve λ-independent factors only, ultimately cancel out in all results involving only spectral determinants.)

The extension of $\psi_\lambda(q)$ to the whole real line, as a solution of equation (16) with the potential $V(|q|)$, is shown by integrations to identically satisfy [11, appendices A and D]

$$D(\lambda) \equiv C W_\lambda \qquad D^{\mathrm{P}}(\lambda) \equiv C^{\mathrm{P}}[-\psi'_\lambda(0)/\psi_\lambda(0)] \tag{42}$$

$$W_\lambda \stackrel{\text{def}}{=} \text{Wronskian}\{\psi_\lambda(-q), \psi_\lambda(q)\} \equiv -2\psi_\lambda(0)\psi'_\lambda(0) \tag{43}$$

for some constants C, C^{P}; to identify these, we test a characteristic property of the spectral determinants: the canonical large-λ asymptotics (equation (8)) of their logarithms down to constants included; as regards the right-hand sides in equation (42), the WKB formulae (39), (41) (good for large λ) supply these asymptotic forms,

$$W_\lambda/2 \sim \mathrm{e}^{2\int_0^{+\infty}\Pi_\lambda(q)\,\mathrm{d}q} \equiv D_{\mathrm{cl}}(\lambda) \qquad -\psi'_\lambda(0)/\psi_\lambda(0) \sim \Pi_\lambda(0) \quad (\lambda \to +\infty) \tag{44}$$

both of which have canonical ($\lambda \to +\infty$) logarithms ($D_{\mathrm{cl}}(\lambda)$ by analogy with $D(\lambda)$, and $\Pi_\lambda(0)$ by inspection); hence, necessarily,

$$D(\lambda) \equiv W_\lambda/2 \equiv -\psi_\lambda(0)\psi'_\lambda(0) \qquad D^{\mathrm{P}}(\lambda) \equiv -\psi'_\lambda(0)/\psi_\lambda(0). \tag{45}$$

Incidentally, the second equation (44) then naturally specifies a 'classical skew determinant', $D^{\mathrm{P}}_{\mathrm{cl}}(\lambda) \stackrel{\text{def}}{=} \Pi_\lambda(0)$, from which classical analogues of $D^{\pm}(\lambda)$ also follow as

$$D^{\pm}_{\mathrm{cl}}(\lambda) \stackrel{\text{def}}{=} \Pi_\lambda(0)^{\pm 1/2} \mathrm{e}^{\int_0^{+\infty}\Pi_\lambda(q)\,\mathrm{d}q}. \tag{46}$$

The main conclusion however concerns the quantum spectral determinants $D^{\pm}$ themselves: upon a straightforward simplification of equation (45), they are expressed in terms of $\psi_\lambda(q)$ by the fundamental identities

$$D^{-}(\lambda) \equiv \psi_\lambda(0) \qquad D^{+}(\lambda) \equiv -\psi'_\lambda(0) \tag{47}$$

(also valid for a rescaled potential, i.e. $V(q) = vq^N + \cdots$).

1.2.3. The main functional relation [9, 10]. Jointly with the original problem (16), its set of 'conjugate' equations is defined by means of complex rotations, as [7, Chapter 2]

$$V^{[\ell]}(q) \stackrel{\text{def}}{=} \mathrm{e}^{-\mathrm{i}\ell\varphi} V(\mathrm{e}^{-\mathrm{i}\ell\varphi/2} q) \qquad \lambda^{[\ell]} \stackrel{\text{def}}{=} \mathrm{e}^{-\mathrm{i}\ell\varphi}\lambda \qquad \varphi \stackrel{\text{def}}{=} \frac{4\pi}{N+2} \tag{48}$$

where $\ell = 0, 1, \ldots, L-1$ [mod L] labels the distinct conjugates, of total number

$$L = N+2 \quad \text{in general} \qquad L = \frac{N}{2} + 1 \quad \text{for even polynomial potentials.} \tag{49}$$

The main result to be used throughout is the *Wronskian identity*, which states a bilinear functional relation between the spectral determinants $D^{\pm}(\lambda)$ and those of the first conjugate equation, namely $D^{[1]\pm}(\mathrm{e}^{-\mathrm{i}\varphi}\lambda)$:

$$\mathrm{e}^{+\mathrm{i}\varphi/4} D^{[1]+}(\mathrm{e}^{-\mathrm{i}\varphi}\lambda) D^{[0]-}(\lambda) - \mathrm{e}^{-\mathrm{i}\varphi/4} D^{[0]+}(\lambda) D^{[1]-}(\mathrm{e}^{-\mathrm{i}\varphi}\lambda) \equiv 2\mathrm{i}\mathrm{e}^{\mathrm{i}\varphi\beta_{-1}(0)/2}. \tag{50}$$

It entails an *exact quantization formula* for the eigenvalues E_k,

$$\frac{2}{\pi} \arg D^{[1]\pm}(-\mathrm{e}^{-\mathrm{i}\varphi}E)_{E=E_k} - \frac{\varphi}{\pi}\beta_{-1}(0) = k + \frac{1}{2} \pm \frac{N-2}{2(N+2)} \qquad \text{for} \quad k = {}^{0,2,4,\ldots}_{1,3,5,\ldots} \tag{51}$$

i.e. this condition determines the spectrum $\{E_k\}$ exactly in terms of the spectrum $\{E_k^{[1]}\}$ of the first conjugate potential $V^{[1]}$, (of which the left-hand side $D^{[1]\pm}(-\mathrm{e}^{-\mathrm{i}\varphi}E)$ are functionals). (Equation (51) together with all its conjugates appear to form a determined system for the resolution of all the spectra $\{E_k^{[\ell]}\}$ at once.)

Here we will often invoke the specific results relating to the homogeneous potentials

$$\hat{H}_N \stackrel{\text{def}}{=} -\frac{\mathrm{d}^2}{\mathrm{d}q^2} + q^N \qquad q \in [0, +\infty) \qquad N \geqslant 1 \text{ integer.} \tag{52}$$

The corresponding formulae are collated in greater detail separately in the appendix, and also in table 1 below for $N = 1, 4$.

1.3. Statement of the problem

The previous exact analysis of section 1.2 is controlled in an essential manner by the degree N of the potential, quite sensibly since q^N defines the most singular interaction term. Still, there are interesting transitional situations where this parameter N can diverge or behave discontinuously while the quantum problem itself has a well defined limit (example: the term q^N is multiplied by a coupling constant $g \to 0$). Earlier studies of such problems show those limits to be very singular, and nonuniform over the energy range. An unexplored challenge is then to control the limiting behaviour of spectral functions, which are symmetric functions involving all eigenvalues simultaneously. The earlier analyses, which essentially work at fixed N, need further development to handle such nonuniform regimes.

Here we will begin to gather some insight about this issue by examining three model problems at varying depths.

2. Exercise 1. Infinite square well as $N \to +\infty$ limit

The infinite square-well potential over the interval $[-1, +1]$ can be realized as the $N \to +\infty$ limit of the homogeneous potential $V(q) \equiv |q|^N$. The solutions of the Schrödinger equation with this potential indeed approach those of the infinite square well, but the limiting behaviours are interestingly singular, and nonuniform with respect to the quantum number k [12].

2.1. The $N = +\infty$ problem

The limiting Schrödinger operator $\hat{H}_\infty$ is given by $V(q) \equiv 0$ in $[-1, +1]$ with Dirichlet boundary conditions at $q = \pm 1$. At infinite N, the spectrum of $|q|^N$ becomes explicit again, $\{E_k = (k+1)^2\pi^2/4\}$, of growth order $\mu_\infty = \frac{1}{2}$. As with finite N, the operator $\hat{H}_\infty$ splits into $\hat{H}^+_\infty$ (over the even eigenfunctions, labelled by even k) and $\hat{H}^-_\infty$ (over the odd eigenfunctions, labelled by odd k).

Here the immediately explicit spectral functions are the *Fredholm* determinants $\Delta^\pm$ of equation (14) (they reduce to standard Weierstrass products),

$$\begin{aligned} &\Delta^+_\infty(\lambda) = \cos\sqrt{-\lambda} \qquad \Delta^-_\infty(\lambda) = \frac{\sin\sqrt{-\lambda}}{\sqrt{-\lambda}} \\ &\Longrightarrow \qquad \Delta_\infty(\lambda) = \frac{\sin 2\sqrt{-\lambda}}{2\sqrt{-\lambda}} \end{aligned} \tag{53}$$

and the plain spectral zeta-functions (related to Riemann's zeta-function $\zeta(s)$),

$$\begin{aligned} &Z^+_\infty(s) = \frac{2^{2s}-1}{\pi^{2s}}\zeta(2s) \qquad Z^-_\infty(s) = \frac{1}{\pi^{2s}}\zeta(2s) \\ &\Longrightarrow \qquad Z_\infty(s) = \left(\frac{2}{\pi}\right)^{2s}\zeta(2s). \end{aligned} \tag{54}$$

The latter formulae imply the explicit computability of $Z^\pm_\infty(s)$ at all integers $s \in \mathbb{Z}$, e.g.

$$\begin{aligned} &Z^+_\infty(0) = 0 \qquad Z^-_\infty(0) = -\tfrac{1}{2} \\ &\Longrightarrow \qquad Z_\infty(0) = -\tfrac{1}{2} \end{aligned} \tag{55}$$

and ultimately that of the spectral determinants themselves (even though these will not serve here) through $D^\pm_\infty(\lambda) = \exp[-(Z^\pm_\infty)'(0)]\Delta^\pm_\infty(\lambda)$, using

$$\exp[-(Z^\pm_\infty)'(0)] \equiv D^\pm_\infty(0) = 2 \qquad \Longrightarrow \qquad D_\infty(0) = 4. \tag{56}$$

2.2. The transitional behaviour problem

Regarding the behaviour of the spectral functions, a first task is to seek conditions ensuring the regular behaviour of a quantity (meaning that it has a finite limit, *and* this is the correct value for the limiting problem). For the other (singular or pathological) quantities then comes the additional task of describing their precise behaviours.

A fundamental quantity in these problems is the growth order μ. Here, the limiting (square-well) value $\mu_\infty = \frac{1}{2}$ agrees with the limit of μ_N $(= \frac{1}{2} + \frac{1}{N}$ by equation (17)); hence μ behaves regularly (as opposed to the later examples).

As a crude dividing line between (generic) singular and regular behaviours, we expect that essentially those quantities which converge both for finite N and in the limiting problem should be regular, in particular

$$Z^\pm_N(s, \lambda) \to Z^\pm_\infty(s, \lambda) \quad \text{iff} \quad \mathrm{Re}\,(s) > \mu_\infty. \tag{57}$$

Singular behaviour should then set in at $\mathrm{Re}\,(s) = \mu_\infty$ and, plausibly, become worse as $\mathrm{Re}\,(s)$ decreases further. This purely qualitative argument cannot, however, predict the precise behaviour of any singular quantity.

Fortunately, quantitative statements are made easier here by a set of explicit results for the finite-N problems (cf appendix A.1) and their counterparts for $N = +\infty$ (section 2.1 above). These data indeed behave consistently with the prediction (57) taken with $\mu_\infty = \frac{1}{2}$. As examples of regular ($s > \mu_\infty$) behaviours, $Z_N^\pm(1) \to Z_\infty^\pm(1) = {}^{1/2}_{1/6}$ by equation (140), and likewise for the higher-order sum rules involving $s = 2, 3, \ldots$ (section 2.5 below). As opposite examples (involving $s < \mu_\infty$): $Z_N(0) \equiv 0$ for all N whereas $Z_\infty(0) = -\frac{1}{2}$ by equations (55), (128); and worse, (136) implies

$$\exp[-(Z_N^\pm)'(s=0)] = D_N^\pm(\lambda = 0) \sim (N/\pi)^{1/2} \to \infty \tag{58}$$

even though $D_\infty^\pm(0)$ have finite values, perfectly defined by equation (56), in the $N = +\infty$ problem! (All this shows how carefully such transitional problems must be handled. The same formulae show the skew spectral functions (20) to behave slightly better: e.g. $Z_N^{\mathrm{P}}(0) \equiv \frac{1}{2} = Z_\infty^{\mathrm{P}}(0)$, $D_N^{\mathrm{P}}(0) \to 1 = D_\infty^{\mathrm{P}}(0)$.)

We then basically expect

$$\Delta_N^\pm(\lambda) \to \Delta_\infty^\pm(\lambda) \qquad \text{but} \quad D_N^\pm(\lambda) \text{ diverge,} \tag{59}$$

this divergence being confined here to the factor $\exp[-Z_N'(0)] = D_N(0)$ alone, because $\mu_\infty < 1$ and the Fredholm determinants can be expressed using $s = 1, 2, \ldots$ only as in equation (14). (The Fredholm determinants $\Delta(\lambda)$ should behave regularly *in general*, since they are designed by retaining only regular values $Z(s)$ (at integers s) in their Taylor series (cf equation (13)); this has to be qualified only if μ reaches (or jumps across) an integer in the limit, as in the example of section 3.)

2.3. The main functional relation

We now study the $N \to +\infty$ limit of the functional relation (50) specialized to homogeneous potentials $|q|^N$, as stated in equation (130). By the preceding arguments, this functional relation should be well behaved as $N \to +\infty$ only once it has been transcribed for Fredholm determinants (using equations (14), (132)):

$$\mathrm{e}^{+\mathrm{i}\varphi/4}\Delta_N^+(\mathrm{e}^{-\mathrm{i}\varphi}\lambda)\Delta_N^-(\lambda) - \mathrm{e}^{-\mathrm{i}\varphi/4}\Delta_N^+(\lambda)\Delta_N^-(\mathrm{e}^{-\mathrm{i}\varphi}\lambda) \equiv 2\mathrm{i}\sin\varphi/4. \tag{60}$$

In the latter formula, holding for all $\varphi = 4\pi/(N+2)$, the expansions of both sides in powers of $\varphi \to 0$ should then be identified order by order. Equation (60) having the form of a 'quantum Wronskian' identity for finite φ [13, 14], it is not surprising that the identification to the leading order $\mathrm{O}(\varphi)$ discloses a 'classical' Wronskian structure:

$$\Delta^+\left(\lambda\frac{\mathrm{d}}{\mathrm{d}\lambda}\Delta^-\right) - \left(\lambda\frac{\mathrm{d}}{\mathrm{d}\lambda}\Delta^+\right)\Delta^- \equiv \frac{1}{2}(1 - \Delta^+\Delta^-). \tag{61}$$

However, we know neither how to interpret the right-hand side, nor how to solve this functional relation directly (and identification at the next order in $\varphi \propto 1/N$ within equation (60) does not appear to yield any new information either): this constitutes an interesting open problem, since the finite-N equation admits constructive solutions by an exact quantization method using equation (133), a method which, however, seems totally singular in the $N \to +\infty$ limit. At the same time, the Fredholm determinants of the infinite square well are known, given by equation (53); they explicitly verify (61), and this provides a positive test of regular $N \to +\infty$ behaviour for the main functional relation in the form (60).

2.4. The $N = \infty$ coboundary and cocycle identities

For the homogeneous finite-N problem, a closed functional equation for the complete determinant $D(\lambda)$ is supplied in the appendix as 'the cocycle identity' (134), a sum of $L = \mathrm{O}(N)$ terms. Its naive $N \to +\infty$ limit will be an integral relation for $D(\lambda)$, further reducible by the residue calculus. It is however simpler to work out the $N \to +\infty$ limit directly upon the (logarithm of the) underlying 'coboundary identity' (131) equivalent to equation (60): this limit is manifestly equivalent to equation (61) and has the (additive) coboundary form

$$-2\lambda \frac{\mathrm{d}}{\mathrm{d}\lambda} \log \Delta^{\mathrm{P}}(\lambda) \equiv \frac{1}{\Delta(\lambda)} - 1. \tag{62}$$

Δ^{P} being meromorphic and Δ entire, the main solvability condition for equation (62) is that the residues at the poles of $1/\Delta$ must match the explicit residues of the left-hand side, resulting in a curious constraint upon Δ alone at its zeros,

$$-2\left[\lambda \frac{\mathrm{d}}{\mathrm{d}\lambda} \Delta(\lambda)\right]_{\lambda=-E_k} = (-1)^k \qquad \text{for all } k \in \mathbb{N} \tag{63}$$

which stands as the $N = \infty$ counterpart for the cocycle identity (134). It is verified by Δ_∞ but, as with equation (61) before, we have no idea about other possible solutions.

2.5. The $N = \infty$ sum rules

The Taylor series of both sides of equation (62) can be expressed with the help of (14), giving

$$2\sum_{n=1}^{\infty} Z^{\mathrm{P}}(n)(-\lambda)^n \equiv \exp\left[\sum_{m=1}^{\infty} \frac{Z(m)}{m}(-\lambda)^m\right] - 1 \equiv \sum_{r=1}^{\infty} \frac{1}{r!}\left[\sum_{m=1}^{\infty} \frac{Z(m)}{m}(-\lambda)^m\right]^r. \tag{64}$$

As in the finite-N case (equation (141)), this acts as a generating identity: the identification of each power λ^n in equation (64) yields a sum rule of order n, which here expresses the combination $2Z^{\mathrm{P}}(n) - Z(n)/n$ in terms of the lower $Z(m)$, as

$$2Z^{\mathrm{P}}(1) - Z(1) = 0 \qquad 2Z^{\mathrm{P}}(2) - Z(2)/2 = Z(1)^2/2 \qquad \text{etc.} \tag{65}$$

The regular behaviour of equation (60) implies that these sum rules too must be the limits of their finite-N counterparts (142) and also be verified by $Z_\infty^{\pm}(s)$. (Due to the special form (54) of $Z_\infty^{\pm}(s)$, these rules amount to equating each Bernoulli number B_{2n} to a certain polynomial in its predecessors.)

In conclusion, this case provides a testing ground for ideas and methods applicable to transitional regimes; however, it has not yet yielded any new results about the underlying spectral problems themselves. Still, we have identified several novel structures in the $N = \infty$ problem as imprints of nontrivial finite-N features in the $N \to \infty$ limit. It remains to effectively handle the finite-N problem as a regular deformation of this $N = \infty$ case, but this would probably require answering the various questions we left open.

3. Exercise 2: Anharmonic perturbation theory as $N = 2$ limit

We now study the approach towards the other singular limit of the formalism, $N = 2$. It cannot be realized through homogeneous polynomials, but a transition from $N = 4$ to 2 precisely underlies the well known perturbation theory for the anharmonic potentials [15–18]

$$U_g(q) = q^2 + gq^4 \ (g \to 0^+) \qquad \text{or equivalently} \qquad V_v(q) = q^4 + vq^2 \ (v \to +\infty) \tag{66}$$

given the basic unitary equivalence between the two operators

$$\hat{H} \stackrel{\text{def}}{=} -\mathrm{d}^2/\mathrm{d}q^2 + V_v \qquad \text{and} \qquad \sqrt{v}(-\mathrm{d}^2/\mathrm{d}q^2 + U_g) \qquad \text{with} \quad g \equiv v^{-3/2}. \tag{67}$$

We denote the v-dependent spectral functions of $\hat{H}^{\pm}$ (+: even, −: odd) as

$$Z^{\pm}(s,\lambda;v) \qquad Z^{\pm}(s;v) \stackrel{\text{def}}{=} Z^{\pm}(s,0;v) \qquad D^{\pm}(\lambda;v). \tag{68}$$

The poles of these zeta-functions lie at $s = 3/4, 1/4, -1/4, \ldots$; moreover, by a straightforward computation of equation (25), $\beta_{-1}(s) \equiv 0$ for an even quartic potential; hence (according to equation (27)) the leading trace identities are v-independent, as

$$Z(0,\lambda;v) \equiv 0 \qquad Z^{\mathrm{P}}(0,\lambda;v) \equiv 1/2 \qquad \text{for all finite } v. \tag{69}$$

3.1. *The transition $v \to \infty$*

This transition is now discontinuous: N retains the fixed value 4 for all finite v while it has the (more singular) value $N = 2$ at $g = 0$. All related parameters are then singular, especially now the order $\mu \equiv \frac{3}{4}$ for all finite v, versus $\mu_\infty = 1$.

Each eigenvalue of $\hat{H}$ satisfies

$$E_k \sim \sqrt{v}(2k+1) \qquad v \to +\infty \tag{70}$$

but not uniformly in k [17], hence it is another matter to find the behaviour of the corresponding spectral determinants $D^{\pm}(\lambda;v)$ themselves, as entire functions of λ and v.

Following equation (57) again, we now expect $Z^{\pm}(s,\lambda;v)$ to be well behaved as $v \to +\infty$ iff $s > \mu_\infty = 1$. For instance, the $s = 0$ trace identity (69) behaves singularly: for the limiting potential $U_{g=0}(q) = q^2$, equation (159) gives

$$Z(0,\lambda) \equiv -\lambda/2 \qquad (\text{but still, } Z^{\mathrm{P}}(0,\lambda) \equiv \tfrac{1}{2}). \tag{71}$$

According to equation (13), $(Z^{\pm})'(0,\lambda;v)$ and $D^{\pm}(\lambda;v)$ should be singular (as previously), but now so should $Z^{\pm}(1,\lambda;v)$ and the resolvent trace $\partial_\lambda \log D^{\pm}(\lambda;v)$ (also involving $s = 1$); thereafter, higher derivatives $(\partial_\lambda)^n \log D^{\pm}(\lambda;v)$ should behave regularly (as they only involve $s = n, n+1, \ldots$). Understanding the $v \to +\infty$ behaviour of $D^{\pm}(\lambda;v)$ then just requires the control of two (pairs of) functions of v alone, $(Z^{\pm})'(0;v)$—or equivalently $D^{\pm}(0;v)$—and $Z^{\pm}(1;v)$.

From now on we will exclusively deal with $D^{\pm}(0;v)$, a problem which entirely resides in the $\{\lambda = 0\}$ plane. It is technically simpler because the anomaly (71) present in (and only in) the limiting problem $U_0(q) = q^2$ vanishes at $\lambda = 0$. Moreover, these restricted spectral functions $D^{\pm}(0;v)$ will display many explicit properties, making them intriguingly similar to spectral determinants of homogeneous potentials. The analysis of the complete spectral determinant $D^{\pm}(\lambda;v)$ (including the coefficient $Z^{\pm}(1;v)$ as $\partial_\lambda \log D^{\pm}(\lambda;v)_{\lambda=0}$) is also under way but will be more involved.

3.2. *'Extraordinairy' spectral functions*

We therefore now specialize to the following pair of restricted determinants:

$$\mathrm{Qi}^{\pm}(v) \stackrel{\text{def}}{=} D^{\pm}(\lambda = 0; v) = \det(-\mathrm{d}^2/\mathrm{d}q^2 + q^4 + vq^2)^{\pm}. \tag{72}$$

These entire functions of v will display numerous explicit properties. (Determinants of general binomial potentials, $\det(-\mathrm{d}^2/\mathrm{d}q^2 + q^N + vq^M)$, $M < N$, can be handled likewise, cf section 4.)

Two immediate results are a pair of special values (cf equation (136) and table 1),

$$\mathrm{Qi}^{+}(0) = D_4^{+}(0) = \frac{6^{1/3}2\sqrt{\pi}}{\Gamma(\frac{1}{6})} \qquad \mathrm{Qi}^{-}(0) = D_4^{-}(0) = \frac{\Gamma(\frac{1}{6})}{6^{1/3}\sqrt{\pi}} \tag{73}$$

Table 1. Analytical and numerical zeta-function values for several spectra. First two columns: Airy zeros (cf appendix A.2.2; notation as in equation (153): $\tau \overset{\text{def}}{=} -\text{Ai}'(0)/\text{Ai}(0) = 3^{1/3}\Gamma(\frac{2}{3})/\Gamma(\frac{1}{3})$). Middle two columns: quartic oscillator ($\hat{H}_4$) levels (cf appendix A.2.1). Last two columns: zeros of $\text{Qi}^{\pm}$ (cf section 3.2). Computations used analytical formulae when available, but were also cross-checked against direct calculations on numerical spectra. Note: $\Gamma(\frac{1}{6}) = 2^{2/3}\sqrt{\pi}\Gamma(\frac{1}{3})/\Gamma(\frac{2}{3}) = 2^{2/3}3^{1/3}\sqrt{\pi}/\tau$.

Spectra:	Airy zeros		Quartic oscillator		Zeros of $\text{Qi}^{\pm}$	
$Z(s)$	Z_1^+	Z_1^-	Z_4^+	Z_4^-	$\mathcal{Z}^+$	$\mathcal{Z}^-$
$Z'(0)$	0.086 1122	−0.229 9537	−0.146 0318	−0.547 1153	−0.168 5422	−0.178 0313
$\text{e}^{-Z'(0)}$	$\frac{2\sqrt{\pi}}{3^{1/3}\Gamma(\frac{1}{3})}$	$\frac{2\sqrt{\pi}}{3^{2/3}\Gamma(\frac{2}{3})}$	$\frac{2^{4/3}3^{1/3}\sqrt{\pi}}{\Gamma(\frac{1}{6})}$	$\frac{2^{2/3}\sqrt{\pi}}{3^{1/3}\Gamma(\frac{5}{6})}$	$\frac{6^{1/3}\Gamma(\frac{1}{4})}{\Gamma(\frac{1}{6})}$	$\frac{\sqrt{2}\Gamma(\frac{1}{6})}{6^{1/3}\Gamma(\frac{1}{4})}$
	$=D_1^+(0)$	$=D_1^-(0)$	$=D_4^+(0)$	$=D_4^-(0)$	$=\mathcal{D}^+(0)$	$=\mathcal{D}^-(0)$
	0.917 4912	1.258 5418	1.157 2330	1.728 2604	1.183 5782	1.194 8628
$Z(0)$	$\frac{1}{4}$	$-\frac{1}{4}$	$\frac{1}{4}$	$-\frac{1}{4}$	$\frac{1}{8}$	$-\frac{1}{8}$
$Z(1)$	0	$-\tau$	$=2Z_4^-(1)$	$\frac{2^{4/3}\pi^3}{3^{17/6}\Gamma(\frac{2}{3})^5}$	$-\frac{3^{4/3}\Gamma(\frac{2}{3})^5}{2^{10/3}\pi^2}$	$=2\mathcal{Z}^+(1)$
		−0.729 0111	1.526 6059	0.763 3029	−0.198 0209	−0.396 0418
$Z(2)$	$1/\tau$	τ^2	cf [11], eqns (C.28, 33, 34)			
	1.371 7212	0.531 4572	0.914 7383	0.081 5825	0.357 8564	0.237 7466
$Z(3)$	1	$-\tau^3 + \frac{1}{2}$				
		0.112 5618	0.841 4950	0.019 0222	0.103 3821	0.038 5889

and a main functional relation, drawn from equation (50) (with $\varphi = 2\pi/3$ and $\beta_{-1} \equiv 0$) and from the conjugacy formula $V^{[\ell]}(q) \equiv q^4 + (\text{j}^\ell v)q^2$ (cf equation (48)):

$$\text{e}^{+\text{i}\pi/6}\text{Qi}^+(\text{j}v)\text{Qi}^-(v) - \text{e}^{-\text{i}\pi/6}\text{Qi}^+(v)\text{Qi}^-(\text{j}v) \equiv 2\text{i} \qquad (\text{j} \overset{\text{def}}{=} \text{e}^{2\text{i}\pi/3}). \tag{74}$$

This functional relation is identical to that of the homogeneous quartic problem (143) if the even and odd arguments are interchanged, otherwise it also resembles the Airy relation (149). Actually, $\text{Qi}^{\pm}$ will display hybrid properties between those two cases (hence their name).

3.2.1. Zeros of $\text{Qi}^{\pm}$. The entire functions $\text{Qi}^{\pm}$ vanish at those values $v = -w_k$ (<0) for which $\lambda = 0$ is an eigenvalue for the potential $q^4 + vq^2$ (*generalized* spectral problem). As before, the labelling of w_k in increasing order makes the parity of k match that of the eigenfunctions. (We do not expect any complex zeros.)

The first few zeros of $\text{Qi}^{\pm}$ evaluate as

$$\begin{array}{cccc} k & (\text{Qi}^+) & k & (\text{Qi}^-) \\ 0 & -2.219\,5971 & 1 & -3.251\,1776 \\ 2 & -5.490\,0693 & 3 & -6.159\,8396 \\ 4 & -7.927\,6920 & 5 & -8.485\,4215 \\ 6 & -10.029\,209 & 7 & -10.525\,121. \end{array} \tag{75}$$

The zeros ($-w_k$) also obey an *exact quantization condition* immediately following from equation (74) by analogy with its quartic (144) and Airy (150) counterparts, of which it looks like a crossbreed:

$$\frac{2}{\pi}\arg \text{Qi}^{\pm}(-\text{j}w)_{w=w_k} = k + \frac{1}{2} \pm \frac{1}{6} \qquad \text{for} \quad k = {}^{0,2,4,\ldots}_{1,3,5,\ldots}. \tag{76}$$

Anticipating the next paragraph on asymptotic results, we can at once estimate the zeros $\{w_k\}$ for large k by solving the Schrödinger equation with potential $V_v(q) = q^4 + vq^2$ and with $\lambda = 0$ semiclassically, for large $v = -w < 0$. For $q \gg 1$, $\psi(q)$ has standard WKB forms in both allowed and forbidden regions (decaying in the latter); at the same time, for $q \ll \sqrt{w}$ (far inside the allowed region), $\psi(q)$ must approximately satisfy $(-\mathrm{d}^2/\mathrm{d}q^2 - wq^2)\psi = 0$, an equation solvable by Bessel functions (cf equation (135)); specifically here,

$$\psi_\pm(q) \propto q^{1/2} J_{\mp 1/4}(\sqrt{w}q^2/2) \quad \left(\begin{smallmatrix}\text{even}\\ \text{odd}\end{smallmatrix} \text{ solutions}\right). \tag{77}$$

Matching the two approximations in the intermediate region $\{1 \ll q \ll \sqrt{w}\}$ then yields the semiclassical quantization formula

$$\begin{aligned} \frac{1}{2\pi} \oint_{p^2+V(q)=0} p\,\mathrm{d}q &\sim k + \frac{3}{4} \qquad \text{for } k \text{ even} \\ &\sim k + \frac{1}{4} \qquad \text{for } k \text{ odd.} \end{aligned} \tag{78}$$

Remark. This Bohr–Sommerfeld quantization rule (78) is appropriate for a *general* symmetric double-well potential with a parabolic barrier top precisely kept at the energy 0; it induces a splitting between even and odd quantized actions exactly half-way between their (quasi) degeneracy towards the bottom of the wells and the equidistant spacing achieved high above the barrier top [19].

Now, for the current potential $V_{-w}(q) = q^4 - wq^2$,

$$\oint_{p^2+V_{-w}(q)=0} p\,\mathrm{d}q = 2\int_{-\sqrt{w}}^{+\sqrt{w}} \sqrt{wq^2 - q^4}\,\mathrm{d}q \equiv \tfrac{4}{3} w^{3/2} \tag{79}$$

giving $\mu = \frac{3}{2}$ as the growth order for this spectrum.

3.2.2. Asymptotic properties. An essential calculation afforded by the present formalism is the asymptotic evaluation of the spectral determinants for $v \to \infty$. The straightforward behaviour (70) of the individual eigenvalues only suggests that $D^\pm(\lambda; v)$ should closely relate to $\det(-\mathrm{d}^2/\mathrm{d}q^2 + vq^2 + \lambda)^\pm \equiv D_2^\pm(\lambda|\sqrt{v})$ (the operator $-\mathrm{d}^2/\mathrm{d}q^2 + vq^2$ being equivalent to $\sqrt{v}\hat{H}_2$). We will therefore need the full expressions of these harmonic spectral determinants for all v, which follow for instance from equations (15), (155), (159):

$$D_2^\pm(\lambda|\sqrt{v}) = \frac{(\sqrt{2}v^{1/8})^{\pm 1 - \lambda/\sqrt{v}}\sqrt{2\pi}}{\Gamma(\frac{2\mp 1+\lambda/\sqrt{v}}{4})}. \tag{80}$$

We will now connect all the determinants through the respective canonical recessive solutions (39), which can be fully evaluated (at $\lambda = 0$) in the WKB approximation: $\psi_{4,\lambda=0}$ for the quartic $V_v(q) = q^4 + vq^2$ on the one hand, for which $\Pi_0(q) = (q^4 + vq^2)^{1/2}$ and

$$\int_q^{+\infty} \Pi_0(q)\,\mathrm{d}q = -\frac{1}{3}(q^2+v)^{3/2} \sim -\frac{q^3}{3} - \frac{v}{2}q + \mathrm{O}\left(\frac{1}{q}\right) \qquad \text{for} \quad q \to +\infty \tag{81}$$

$$\Longrightarrow \qquad \psi_{4,0} \sim (q^4 + vq^2)^{-1/4} \mathrm{e}^{-(q^2+v)^{3/2}/3} \qquad \text{for} \quad (q^4 + vq^2) \to +\infty \tag{82}$$

and $\psi_{2,\lambda=0}$ for the harmonic potential (vq^2) on the other hand, for which $\Pi_0(q) = v^{1/2}q$ and

$$\begin{aligned} &\int_q^{+\infty} \Pi_0(q)\,\mathrm{d}q = -\frac{\sqrt{v}}{2}q^2 \\ &\Longrightarrow \qquad \psi_{2,0} \sim (vq^2)^{-1/4} \mathrm{e}^{-\sqrt{v}q^2/2} \qquad \text{for} \quad (vq^2) \to +\infty. \end{aligned} \tag{83}$$

(In both cases, the large-q behaviour was checked against equation (33).)

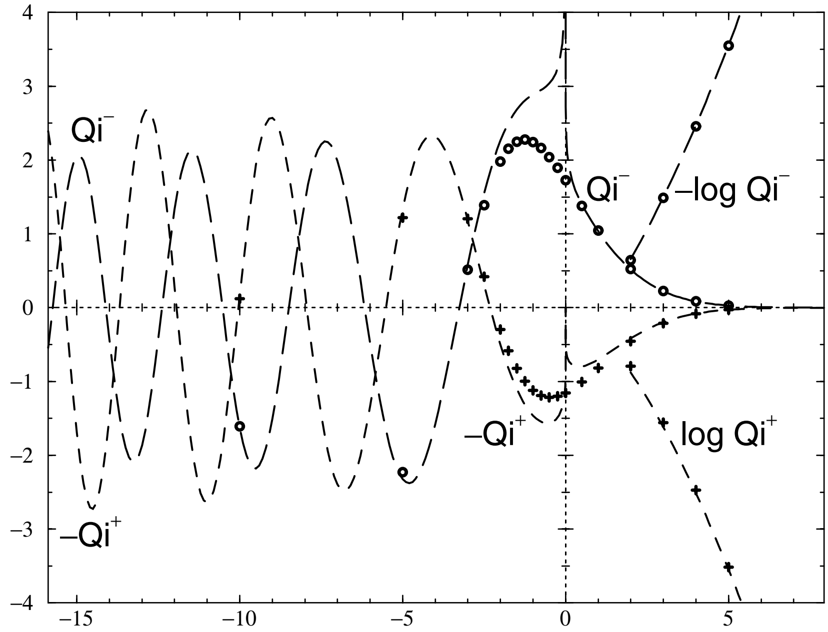

Figure 1. The functions Qi^- and $(-\mathrm{Qi}^+)$ (of equation (72)), which specially resemble the Airy functions Ai and Ai′ respectively, plus logarithmic plots to magnify their large-v asymptotics on the decaying side ($v \gg 0$). Numerical values: + for Qi^+, and ∘ for Qi^- (computed using equation (38), with an input of $K \approx 10^3$ numerical eigenvalues of the operator $(-\mathrm{d}^2/\mathrm{d}q^2 + q^4 + vq^2)$ for every value of the coupling constant v). The dashed lines plot the large-v asymptotic formulae (86), (87).

Now comes the central feature when $v \to \infty$: while the quartic recessive solution $\psi_{4,0}(q)$ (initially specified when $q \to +\infty$) has to match a harmonic recessive solution upon penetrating the intermediate region $1 \ll q \ll \sqrt{v}$, the *normalization* of $\psi_{4,0}$ (canonically set for $q \gg \sqrt{v}$) need *not* match that of $\psi_{2,0}$ (canonically set for $1 \ll q$ and no q^4 term). A crucial quantity is actually the ratio $(\psi_{4,0}/\psi_{2,0})$, and it simply emerges by reexpanding equation (82) for $\sqrt{v} \gg q$, as

$$\psi_{4,0} \sim (vq^2)^{-1/4}\mathrm{e}^{-v^{3/2}/3-\sqrt{v}q^2/2} \sim \mathrm{e}^{-v^{3/2}/3}\psi_{2,0} \qquad \text{for} \quad 1 \ll q \ll \sqrt{v}. \quad (84)$$

Thereupon, invoking equation (47) once for the quartic case, then for the harmonic case at $\lambda = 0$, and finally equation (80), the latest result translates to

$$D^{\pm}(0; v) \sim \mathrm{e}^{-v^{3/2}/3} D_2^{\pm}(0|\sqrt{v}) \sim \mathrm{e}^{-v^{3/2}/3} v^{\pm 1/8} D_2^{\pm}(0) \qquad (85)$$

that is,

$$\begin{aligned} \mathrm{Qi}^+(v) &\sim \frac{2\sqrt{\pi}}{\Gamma(\frac{1}{4})} v^{+1/8}\mathrm{e}^{-v^{3/2}/3} \\ \mathrm{Qi}^-(v) &\sim \frac{\sqrt{\pi}}{\Gamma(\frac{3}{4})} v^{-1/8}\mathrm{e}^{-v^{3/2}/3} \quad (v \to \infty). \end{aligned} \qquad (86)$$

So, we obtained asymptotic behaviours for these new functions which strongly resemble those of the Airy functions ($-\mathrm{Ai}'$ and Ai respectively), with an identical growth order $\mu = \frac{3}{2}$. Similar reasonings should extend equation (86) to complex v with $|\arg v| < \pi$, and to

$$\begin{aligned} \mathrm{Qi}^{+}(-w) &\sim \frac{4\sqrt{\pi}}{\Gamma(\frac{1}{4})} w^{+1/8} \cos\left[\frac{w^{3/2}}{3} + \frac{\pi}{8}\right] \\ \mathrm{Qi}^{-}(-w) &\sim \frac{2\sqrt{\pi}}{\Gamma(\frac{3}{4})} w^{-1/8} \cos\left[\frac{w^{3/2}}{3} - \frac{\pi}{8}\right] \end{aligned} \tag{87}$$

for $w \to +\infty$.

The asymptotic formula (78), (79) for the zeros $(-w_k)$ can now be consistently regained: either from the asymptotic formula (87) on the negative real axis, or from the exact quantization condition (76) asymptotically expanded by means of (86) on the half-line $\{\arg v = \pi/3\}$.

Figure 1 plots the pair of functions $\mathrm{Qi}^{\pm}$ and their asymptotic forms. Given their variegated properties, the idea that $\mathrm{Qi}^{\pm}$ should be reducible to simpler known functions seems unlikely.

3.2.3. Spectral functions of the zeros and further exact results. We must also consider spectral functions of the generalized spectrum $\{w_k\}$: its zeta-functions $\mathcal{Z}^{\pm}(s, v)$, and spectral determinants $\mathcal{D}^{\pm}(v)$, which now refer to a singular operator, i.e. (with $\hat{H}_4 = -\mathrm{d}^2/\mathrm{d}q^2 + q^4$, cf (52))

$$\begin{aligned} \mathcal{Z}^{\pm}(s, v) &\equiv \mathrm{Tr}\,[(q^{-1}\hat{H}_4 q^{-1} + v)^{\pm}]^{-s} \\ \mathcal{D}^{\pm}(v) &\equiv \det(q^{-1}\hat{H}_4 q^{-1} + v)^{\pm} \end{aligned} \tag{88}$$

whereas $\mathrm{Qi}^{\pm}(v) = \det(\hat{H}_4 + vq^2)^{\pm}$; hence we cannot readily assert that this spectrum $\{w_k\}$ is admissible in the sense of section 1, nevertheless all the ensuing consequences are numerically verifiable and support such an assumption. (If this singular operator $q^{-1}\hat{H}_4 q^{-1}$ is self-adjoint, then the generalized spectrum $\{w_k\}$ is real as we assumed.)

Firstly, the semiclassical quantization conditions (78), (79) for this spectrum fix the leading trace identities according to equation (36):

$$b_0^{\pm} = \mp\tfrac{1}{4} \qquad \Longrightarrow \qquad \mathcal{Z}^{\pm}(0) = \pm\tfrac{1}{8}. \tag{89}$$

Then, even though $\mathrm{Qi}^{\pm}(v)$ are spectral determinants (at a frozen energy), they do not have to coincide with $\mathcal{D}^{\pm}(v)$. Simply, both being entire functions with the same order $\frac{3}{2}$ and the same zeros, they must be related as $\mathcal{D}^{\pm}(v) \equiv C^{\pm}\mathrm{e}^{c^{\pm}v}\mathrm{Qi}^{\pm}(v)$ ($C^{\pm}$, $c^{\pm}$ constants).

Quite generally, a complete identification of such free constants can be based on a principle of 'semiclassical compliance' whenever the spectrum of zeros is admissible: the spectral determinant is *a priori* known up to a factor $\exp P(\lambda)$ (with P a polynomial of degree $\leqslant \mu =$ the growth order), but only for one such P can the *canonical semiclassical form* (8) be also satisfied, and this lifts the ambiguity completely.

Here, the asymptotic formulae (85) are known for $\log \mathrm{Qi}^{\pm}(v)$ down to the constant terms (included), and only the latter are actually noncanonical in $v \to +\infty$, hence necessarily

$$\mathcal{D}^{\pm}(v) \equiv \mathrm{Qi}^{\pm}(v)/D_2^{\pm}(0). \tag{90}$$

Several results follow from equation (90) (and (73)):

- Explicit *Stirling constants* for the spectrum $\{w_k\}$ (playing the same role as $\sqrt{2\pi}$ for the integers, in view of equation (38) at $\lambda = 0$):

$$\mathcal{D}^{\pm}(0) \equiv \exp[-\mathcal{Z}^{\pm\prime}(0)] = D_4^{\pm}(0)/D_2^{\pm}(0) \qquad \text{(see table 1).} \tag{91}$$

- Upon a simple explicit rescaling, the expansions (13) written for $\mathcal{D}^{\pm}(v)$ yield

$$\mathrm{Qi}^{\pm}(v) = \mathrm{Qi}^{\pm}(0)\exp\left[-\sum_{n=1}^{\infty}\frac{\mathcal{Z}^{\pm}(n)}{n}(-v)^n\right] \qquad (|v| < w_0). \tag{92}$$

Equations (91), (92) also amount to

$$[-\log \mathrm{Qi}^{\pm}]^{(n)}(0) = \begin{cases} (-1)^n (n-1)! \mathcal{Z}^{\pm}(n) & (n \neq 0) \\ (\mathcal{Z}^{\pm})'(0) + (Z_2^{\pm})'(0) & (n = 0). \end{cases} \tag{93}$$

With these results, we can now draw further consequences from the functional relation (74) and from its coincidence with the homogeneous quartic equation (143) up to the exchange of parities.

- The complete determinants $D = D^+ D^-$ being unaffected by this interchange, the quartic cocycle identity (145) of D_4 must remain satisfied by its analogue, namely the new product function $(\mathrm{Qi}^+ \mathrm{Qi}^-)$. Hence that functional equation (145) now shows *two completely different* entire solutions, the former having order $\mu_4 = \frac{3}{4}$ and the latter $\mu_1 = \frac{3}{2}$ (tied by the duality relation $\mu_4^{-1} + \mu_1^{-1} = 2$). (We do not know if still other nontrivial entire solutions may exist.)
- Because of equations (90)–(92), it is now the spectral zeta-functions $\mathcal{Z}^{\pm}(s)$ which inherit sum rules for $s = n \geqslant 1$, and these have to be the quartic rules (146) with even/odd arguments swapped, giving

$$\begin{aligned} &\mathcal{Z}^-(1) - 2\mathcal{Z}^+(1) = 0 \\ &2\mathcal{Z}^-(2) - \mathcal{Z}^+(2) = 3[\mathcal{Z}^-(1) - \mathcal{Z}^+(1)]^2 \\ &\mathcal{Z}(3) = \mathcal{Z}(1)^3/6 - \mathcal{Z}(1)\mathcal{Z}(2)/2 \qquad \text{etc} \end{aligned} \tag{94}$$

(every identity of order $3n$ expresses $\mathcal{Z}(3n)$ in terms of the lower $\mathcal{Z}(m)$ in exactly the same form as for the quartic zeta-value $Z_4(3n)$, cf equation (146)).

We finally evaluate $\mathcal{Z}^{\pm}(1)$ in closed form, by analogy with the derivation of equation (140) for $Z_N^{\pm}(1)$. The values $\mathcal{Z}^{\pm}(1)$ themselves are regularized quantities, but the first sum rule (94) also implies $-\mathcal{Z}^+(1) = \mathcal{Z}^{\mathrm{P}}(1)$, and the latter has the (semi) convergent defining series $\sum_{k=0}^{\infty}(-1)^k/w_k$. This series can actually be summed when the w_k are more generally defined (for $N \geqslant 3$) as the roots of $\det(-\mathrm{d}^2/\mathrm{d}q^2 + |q|^N - wq^2) = 0$, i.e. as the eigenvalues of the singular operator $q^{-1}\hat{H}_N q^{-1}$ (cf equation (88)); hence

$$\mathcal{Z}_N^{\mathrm{P}}(1) = \mathrm{Tr}\, \hat{P}(q^{-1}\hat{H}_N q^{-1})^{-1} = \mathrm{Tr}\, \hat{P} q \hat{H}_N^{-1} q \qquad \hat{P} = \text{ parity operator.} \tag{95}$$

Then, thanks to the explicit formulae (135), (137) giving the kernel of $\hat{H}_N^{-1}$, and in full parallel with equation (139), this turns into the explicit integral

$$\mathcal{Z}_N^{\mathrm{P}}(1) = \frac{4\nu}{\pi} \sin \nu\pi \int_0^{\infty} [K_\nu(2\nu q^{1+N/2})]^2 q^3 \, \mathrm{d}q \qquad \nu = \frac{1}{N+2} \tag{96}$$

i.e. a (convergent) Weber–Schafheitlin integral, which finally gives

$$\mathcal{Z}_N^{\mathrm{P}}(1) = \frac{\sin \nu\pi}{2\sqrt{\pi}} (2\nu)^{2-8\nu} \frac{\Gamma(3\nu)\Gamma(4\nu)\Gamma(5\nu)}{\Gamma(4\nu + \frac{1}{2})}. \tag{97}$$

For $N = 4$, by the first of equations (94), this also yields the special values $-\mathcal{Z}^+(1) = -\frac{1}{2}\mathcal{Z}^-(1)$ as given in table 1 ($= -(\log \mathrm{Qi}^+)'(0) = -\frac{1}{2}(\log \mathrm{Qi}^-)'(0)$ as well, thanks to equation (93) for $n = 1$).

Table 1 includes some values of these zeta-functions $\mathcal{Z}^{\pm}(s)$. We computed them both analytically and by brute force (using the data from equations (75), (78), (79) within equations (35), (38)), and were thus able to numerically check all the above results.

3.3. A mirror problem without a solution

The new functions $\mathrm{Qi}^{\pm}$ algebraically resemble the spectral determinants of the homogeneous quartic potential, but share many qualitative and asymptotic properties with the Airy functions, especially their order $\mu_1 = \frac{3}{2}$. This coincidence clearly reflects the $N = 1 \leftrightarrow 4$ duality. It is then tempting to seek a fourth pair of functions in the unoccupied symmetrical position: verifying the Airy functional identities (with +/− arguments swapped), but having the order $\mu_4 = \frac{3}{4}$ and the qualitative features of the homogeneous quartic determinants.

However, not only is it difficult to conceive such functions around the spectral framework of the linear potential, but in fact it is easily shown that a perfect mirror solution cannot exist. The functions of such a pair should have only negative zeros $(-\varepsilon_k)$ to fully resemble the determinants $D_4^{\pm}$. They would also obey the Airy sum rules (152) with the +/− superscripts exchanged, i.e.

$$Z^{-}(1) = 0 \qquad Z^{+}(2) = Z^{+}(1)^2 \qquad \text{etc} \tag{98}$$

with $Z^{\pm}(s) = \sum_k \varepsilon_k^{-s}$ (running over ${}^{\text{even}}_{\text{odd}}\, k$) now also for $s = 1$ since $\mu = \frac{3}{4} < 1$ (in contrast to the Airy case, where $Z^{\pm}(1)$ underwent regularization). Under this precise circumstance, each of the leading putative sum rules above is already impossible to satisfy. Therefore, we cannot obtain other partner functions to $\mathrm{Qi}^{\pm}$ in the precise manner described.

(By contrast, this question remains open if we relax the ε_k to allow negative or complex values, or the order constraint $\mu = \frac{3}{4}$.)

3.4. Concluding remarks

Our study of the functions $\mathrm{Qi}^{\pm}$ has remained introductory. We have explicitly proved neither that their zeros are purely real (negative), nor that they form an admissible sequence. It would also be nice to know their asymptotic expansions (86), (87) to all orders in v as for the Airy function (these would also give higher trace identities for $\mathcal{Z}^{\pm}(-m)$); and accessorily, to find closed forms for $\mathcal{Z}^{\pm}(2)$ like those for $Z_N^{\pm}(2)$ [11, appendix C]. (From a general standpoint, one needs to extend the formalism of section 1 to operators $\hat{H}$ which can be *singular* as in equation (88), so as to encompass the $\mathrm{Qi}^{\pm}$ functions; self-adjointness of these operators should also be ascertained.)

The functions $\mathrm{Qi}^{\pm}$ have truly revealed hybrid features. Their analytical and algebraic properties are undoubtedly quartic, while their asymptotic properties are close to the Airy case. (Unlike the Airy functions, they do not satisfy any obvious differential, or linear-difference, equations.) All in all, their structure is very simple and strongly reminiscent of the spectral determinants of homogeneous potentials (cf the appendix), but overall not reducible to the latter; thus, $\mathrm{Qi}^{\pm}$ provide seemingly new solutions to the functional identities governing the homogeneous quartic determinants. We therefore hope that they might also find some roles in the correspondences recently unravelled between those functional equations and exactly solvable models of statistical mechanics or conformal field theory [14].

4. Exercise 3. Quasi-exactly solvable binomial potentials

Keeping the same techniques as previously initiated for the functions $\mathrm{Qi}^{\pm}$, we now turn to the zero-energy determinants for some other binomial potentials on the half-line $\{q \geqslant 0\}$: namely,

$$V(q) = q^N + vq^M \qquad \text{with } N \text{ even and } M \equiv \frac{N}{2} - 1 \text{ throughout} \tag{99}$$

because the corresponding Schrödinger equation (16) has special properties: it is solvable at $\lambda = 0$ (in terms of confluent hypergeometric functions [20, 21]), and for selected values of v it also provides the simplest examples of *quasi-exactly solvable* systems [22]. Then all calculations may strictly follow the previous pattern (referring to section 3 for details), yet some of the final results will be quite different.

For the exact formalism of section 1, retaining the notations of equation (48), these potentials enjoy an exclusive symmetry, namely, all their conjugate potentials are *real*:

$$V^{[\ell]}(q) \equiv q^N + (-1)^\ell v q^M \qquad \text{for all } \ell. \tag{100}$$

Moreover, the evaluation of equation (25) gives a special nonzero residue formula for the first time here: $(V(q)+\lambda)^{-s+\frac{1}{2}} \sim q^{N(-s+\frac{1}{2})} + v(-s+\frac{1}{2})q^{-1-Ns}$ for $q \to +\infty$ implies

$$\beta_{-1}(s) \equiv v(-s+\tfrac{1}{2}) \qquad \text{(independent of } \lambda, N). \tag{101}$$

We henceforth focus upon the restricted determinants $D_N^\pm(\lambda = 0; v)$, where

$$D_N^\pm(\lambda; v) \stackrel{\text{def}}{=} \det(-\mathrm{d}^2/\mathrm{d}q^2 + q^N + vq^{\frac{N}{2}-1} + \lambda)^\pm. \tag{102}$$

As with $\mathrm{Qi}^\pm$ before, two immediate results are a pair of special values explicitly recoverable from equation (136),

$$D_N^\pm(0;0) = \det(-\mathrm{d}^2/\mathrm{d}q^2 + q^N)^\pm = D_N^\pm(0) \tag{103}$$

and a main functional relation, drawn from equation (50) but now taking a special form, due to the particular dependence of the exponent M upon N and to equation (101) (we recall that $\varphi = \frac{4\pi}{N+2}$):

$$\mathrm{e}^{+\mathrm{i}\varphi/4} D_N^+(0;-v) D_N^-(0;v) - \mathrm{e}^{-\mathrm{i}\varphi/4} D_N^+(0;v) D_N^-(0;-v) \equiv 2\mathrm{i}\mathrm{e}^{+\mathrm{i}\varphi v/4}. \tag{104}$$

4.1. Zeros of $D_N^\pm(0; v)$

The zeros of $D_N^\pm(0; v)$ are again the values $v = -w_k < 0$ for which $\lambda = 0$ is an eigenvalue of $\hat{H}_N + v|q|^M$ (a generalized spectral problem). But now equation (104) is very close to the harmonic functional relation (156), especially considering the value π of the rotation angle acting on the spectral variable (and also the special right-hand side phase); then just like equation (156), equation (104) splits into real and imaginary parts and reduces to

$$D_N^+(0;v) D_N^-(0;-v) \equiv 2\left(\sin\frac{\varphi}{2}\right)^{-1} \cos\frac{\varphi}{4}(v-1) \tag{105}$$

which is exactly a (shifted) *gamma-function reflection formula*. This spectral problem is then exactly solvable like the harmonic case: the zeros of the right-hand side, which form one doubly infinite arithmetic progression, must simply be dispatched according to their signs towards one or the other factor on the left-hand side, resulting in the *exact* eigenvalue formulae

$$w_{2n} = \frac{N}{2} + (N+2)n \qquad w_{2n+1} = \frac{N}{2} + 2 + (N+2)n \tag{106}$$

$$\text{i.e.}\ \frac{2}{N+2} w_k = k + \frac{1}{2} \pm \frac{N-2}{2(N+2)} \qquad \text{for} \quad k = {}^{0,2,4,\ldots}_{1,3,5,\ldots}. \tag{107}$$

The spectrum $\{w_k\}$ therefore has growth order *unity*, irrespective of N. For $N = 2$, the generalized spectral problem restores the standard harmonic oscillator problem $\det(-\mathrm{d}^2/\mathrm{d}q^2 + q^2 + v) = 0$. In reverse, the subsequent results will prove perfect generalizations to all even degrees N of the classic harmonic oscillator properties. This clearly provides another view on the partial solvability properties of the potentials (99).

For instance, we can show that *semiclassical quantization is exact* for this zero-energy generalized spectrum. We proceed just as for $\mathrm{Qi}^{\pm}$ in section 3.2.1; for large $v = -w < 0$, now the comparison equation is $(-\mathrm{d}^2/\mathrm{d}q^2 - wq^M)\psi = 0$, solved by

$$\psi_{\pm}(q) \propto q^{1/2} J_{\mp 2\nu}\left(\sqrt{w}\frac{q^{\frac{N}{4}+\frac{1}{2}}}{\frac{N}{4}+\frac{1}{2}}\right) \qquad \left({}^{\text{even}}_{\text{odd}} \text{ solutions}\right) \qquad \nu \stackrel{\text{def}}{=} \frac{1}{N+2} \tag{108}$$

hence the asymptotic matching in the intermediate region $1 \ll q \ll w^{2\nu}$ yields the semiclassical quantization formula

$$\frac{1}{2\pi}\oint_{p^2+V(q)=0} p\,\mathrm{d}q \sim k + \frac{1}{2} \pm \frac{N-2}{2(N+2)} \qquad \text{for} \quad k \;{}^{\text{even}}_{\text{odd}} \tag{109}$$

while the left-hand side, under the change of variables $wx = q^{\frac{N}{2}+1}$, yields

$$\frac{4}{2\pi}\int_0^{w^{2\nu}} (wq^M - q^N)^{1/2}\,\mathrm{d}q = \frac{4w}{(N+2)\pi}\int_0^1 \sqrt{\frac{1-x}{x}}\,\mathrm{d}x = \frac{2}{N+2}w \tag{110}$$

thus the resulting Bohr–Sommerfeld rule manifestly coincides with the exact one (107).

The exact quantization formula (107)—like our earlier equation (76) for the zeros of $\mathrm{Qi}^{\pm}(v) = \det(-\mathrm{d}^2/\mathrm{d}q^2 + q^4 + vq^2)^{\pm}$—displays a hybrid character. The right-hand side of equation (107) is that of the exact quantization formula (51) for an *arbitrary* potential of degree N (whose semiclassical quantization formula is definitely different); whereas its left-hand side is *linear* in the spectral variable (entirely originating from the term $(-\varphi\beta_{-1}(0)/\pi)$ in equation (51)), *and* its semiclassical form is *exact*, both as in the harmonic case.

4.2. Asymptotic properties

We can obtain the asymptotic $v \to +\infty$ form of $D_N^{\pm}(0; v)$ just as in section 3.2.2: we relate these determinants to the canonical recessive solution Ψ_N for the potential $V(q) = q^N + vq^M$ at $\lambda = 0$, then match Ψ_N with the analogous solution ψ_M for the comparison potential vq^M. Hence we need the canonical WKB forms (39) for these two potentials, and primarily the symbolic integral $\int_q^{\infty} V(q')^{1/2}\,\mathrm{d}q'$.

- Full potential V: with a change of variables as above, $\int V(q)^{1/2}\,\mathrm{d}q = 4\nu v \int \sqrt{1+x^2}\,\mathrm{d}x$, we have the obvious primitive

$$\int_0^q V(q')^{1/2}\mathrm{d}q' = 2\nu\left[v \,\mathrm{arcsinh}\frac{q^{\frac{N}{4}+\frac{1}{2}}}{\sqrt{v}} + q^{\frac{N}{4}+\frac{1}{2}}(v + q^{\frac{N}{2}+1})^{1/2}\right]$$
$$\sim 2\nu\left[v\left(\left(\frac{N}{4}+\frac{1}{2}\right)\log q - \frac{1}{2}\log v + \log 2\right) + q^{\frac{N}{2}+1} + \frac{v}{2}\right] \quad (q \to +\infty). \tag{111}$$

We must then set $\int_q^{+\infty} V(q')^{1/2}\mathrm{d}q' \stackrel{\text{def}}{=} -\int_0^q V(q')^{1/2}\mathrm{d}q' + C$, with the target that the large-q behaviour of $\Psi_N \sim V(q)^{-1/4}\exp[\int_q^{+\infty} V(q')^{1/2}\,\mathrm{d}q']$ should obey the canonical equation (40), now with $\mathrm{e}^{\mathcal{C}} = 2^{-v/N}$ by equations (33), (101); i.e.

$$\Psi_N(q) \sim 2^{-v/N} q^{-N/4-v/2}\mathrm{e}^{-2\nu q^{1+N/2}}. \tag{112}$$

This adjustment for C according to equations (111), (112) yields the full WKB specification of the canonical recessive solution, when $V(q) \equiv q^N + vq^M$, as

$$\Psi_N(q) \sim 2^{-v/N}\mathrm{e}^{-\nu v(\log v - 1 - 2\log 2)}V(q)^{-1/4} \times \exp\left[-\int_0^q V(q')^{1/2}\,\mathrm{d}q'\right] \qquad (V(q) \to +\infty). \tag{113}$$

- Comparison with the potential vq^M: the corresponding canonical recessive solution ψ_M is expressible from equation (135) exactly, but its WKB form suffices here:

$$\psi_M(q) \sim (vq^M)^{-1/4} \exp\left[-\sqrt{v} q^{\frac{M}{2}+1} \Big/ \left(\frac{M}{2}+1\right)\right] \tag{114}$$

Now, the matching with equations (111), (113) re-expanded for $v \gg q$ yields the result

$$\Psi_N(q) \sim 2^{-v/N} \mathrm{e}^{-\nu v(\log v - 1 - 2\log 2)} \psi_M(q) \qquad (v \to +\infty). \tag{115}$$

This then translates back to the determinants, by equation (47), as

$$\det\left(-\frac{\mathrm{d}^2}{\mathrm{d}q^2} + q^N + vq^M\right)^{\pm} \sim 2^{-v/N} \mathrm{e}^{-\nu v(\log v - 1 - 2\log 2)} \det\left(-\frac{\mathrm{d}^2}{\mathrm{d}q^2} + vq^M\right)^{\pm}. \tag{116}$$

However, $\det(-\mathrm{d}^2/\mathrm{d}q^2 + vq^M)^{\pm} \equiv v^{4\nu Z_M^{\pm}(0)} D_M^{\pm}(0)$ by equation (15), so, finally,

$$D_N^{\pm}(0; v) \sim \mathrm{e}^{-\frac{1}{N+2} v(\log v - 1)} 2^{\frac{N-2}{N(N+2)} v} v^{\pm\frac{1}{N+2}} D^{\pm}_{\frac{N}{2}-1}(0) \qquad v \to +\infty. \tag{117}$$

4.3. Spectral functions of the zeros and further exact results

As in section 3.2.3, we also need the spectral determinants $\mathcal{D}_N^{\pm}(v)$ built directly for the generalized (even and odd) spectra $\{w_k\}$. Since these form the exact (semi-infinite) arithmetic progressions (107), of growth order 1, the answer must now have the exact form $\mathcal{D}_N^{\pm}(v) \equiv C^{\pm}\mathrm{e}^{c^{\pm}v}/\Gamma(\nu(v \mp 1) + \frac{1}{2})$. Compliance with equation (8*a*, *c*) then fixes the constants, giving

$$\mathcal{D}_N^{\pm}(v) \equiv \frac{\nu^{\nu(v\mp 1)}\sqrt{2\pi}}{\Gamma(\nu(v \mp 1) + \frac{1}{2})} \tag{118}$$

(which agrees with the harmonic spectral determinants (155) for $N = 2$).

We can now complete the evaluation of the former determinants $D_N^{\pm}(0; v)$ themselves, which must have the same zeros as $\mathcal{D}_N^{\pm}(v)$ and the same order unity, hence necessarily $D_N^{\pm}(0; v) \equiv C_{\pm}\mathrm{e}^{c_{\pm}v}\mathcal{D}_N^{\pm}(v)$. This plus the semiclassical constraint (8*a*, *c*) again fixes the constants, yielding

$$D_N^{\pm}(0; v) \equiv 2^{\frac{N-2}{N(N+2)} v} D^{\pm}_{\frac{N}{2}-1}(0)\mathcal{D}_N^{\pm}(v). \tag{119}$$

In contrast to equation (90) for the Qi$^{\pm}$ case, here the latter determinants are known by equation (118), so we end up with *fully closed forms*,

$$\begin{aligned} D_N^{+}(0; v) &\equiv -\frac{2^{-v/N}(4\nu)^{\nu(v+1)+\frac{1}{2}}\Gamma(-2\nu)}{\Gamma(\nu(v-1)+\frac{1}{2})} \\ D_N^{-}(0; v) &\equiv \frac{2^{-v/N}(4\nu)^{\nu(v-1)+\frac{1}{2}}\Gamma(2\nu)}{\Gamma(\nu(v+1)+\frac{1}{2})}. \end{aligned} \tag{120}$$

Remark. At $v = 0$, the identity (119) specifying the ratios $C_{\pm}\mathrm{e}^{c_{\pm}v} = D_N^{\pm}(0; v)/\mathcal{D}_N^{\pm}(v)$ simply becomes the duplication formula for $\Gamma(2\nu)$, a fact which also directly fixes the constants $C_{\pm}$; by contrast, we see no 'cheap' way to obtain the constants $c_{\pm}$—i.e. $c_+ = c_- = (\log 2)(2\nu - \frac{1}{N}) = (\log 2)(\frac{N-2}{2(N+2)})$; in particular, equation (105) alone is of no avail in this respect.

Alternatively, we can invoke the solvability of this potential at zero energy to obtain the spectral determinants directly, just as we deduced equation (136) for the homogeneous potentials (cf appendix). Here the canonical recessive solution Ψ_N normalized by equation (112) can be expressed in terms of a confluent hypergeometric function $U(a, b, z)$ or equivalently a Whittaker function $W_{\kappa,\nu}(z)$, as [23, equation 2.273(12)]

$$\begin{aligned}\Psi_N(q) &\equiv 2^{-v/N}(4\nu)^{\nu(v-1)+\frac{1}{2}}\mathrm{e}^{-2\nu q^{1+N/2}}U(\nu(v-1)+\tfrac{1}{2}, 1-2\nu, 4\nu q^{1+N/2}) \\ &\equiv 2^{-v/N}(4\nu)^{\nu v}q^{-N/4}W_{-\nu v,\nu}(4\nu q^{1+N/2})\end{aligned} \qquad (121)$$

(the normalization is fixed by reference to the known $q \to +\infty$ forms of these functions). In turn, the connection formula

$$U(a, b, z) \equiv \frac{\Gamma(1-b)}{\Gamma(1+a-b)}M(a, b, z) + z^{1-b}\frac{\Gamma(b-1)}{\Gamma(a)}M(1+a-b, 2-b, z)$$

together with $M(a, b, 0) \equiv 1$, determines the values of $\Psi_N(0)$, $\Psi'_N(0)$, which finally yield the determinants through equation (47): these then coincide with equation (120) indeed. (Conversely, the present method can be viewed as a purely spectral derivation of the connection formula for that confluent hypergeometric function. Again, however, the right reference normalization (112) of Ψ_N has to be expressly fed in.)

Remark. The knowledge of $\Psi_N(q)$ also yields an expression for $\sum(-1)^k/w_k$, in full analogy with equations (95)–(97) above, but here this only recovers a special case of the known integrals $\int_0^\infty W_{\kappa,\nu}(z)^2\mathrm{d}z/z$ [31, equation 7.611(4)]; likewise, the sum rules for $\sum(\pm1)^k/w_k^n$ $(n > 1)$ should only reproduce elementary identities among Hurwitz zeta-values here.

For $N = 2$, equation (120) restores the ordinary harmonic spectral determinants (155). The next even case, $V(q) = q^6 + vq^2$, is strongly highlighted in the studies on quasi-exactly solvable potentials [22]. However, our present results hold identically irrespective of the parity of the full potential $q^N + vq^M$, beginning with the potential $q^4 + vq$ on the half-line $\{q > 0\}$, so we end with a few remarks about these noneven potentials.

4.4. The case of noneven potentials

Our exact results above hold equally well for noneven potentials $V(q) = q^N + vq^{\frac{N}{2}-1}$, obtained when N is a multiple of 4, as for even ones. Then, as always, the exact zeros $\{w_k\}$ of $\det(-\mathrm{d}^2/\mathrm{d}q^2 + q^N + vq^{\frac{N}{2}-1})^\pm$ as given by equation (107) refer to the potential defined on the half-line $\{q > 0\}$, with a Neumann/Dirichlet condition at $q = 0$ for the +/− parity, or equivalently to the singular potential $V(|q|)$ over the whole real line.

For N a multiple of 4, an additional exact spectral property is derivable. We now consider the (noneven) potential $V(q)$ over the whole real line. The complete spectral determinant $\boldsymbol{D}$ for such a potential of even degree N is not given by $D(\lambda) = D^+(\lambda)D^-(\lambda)$ (which corresponds to the even potential $V(|q|)$), but in full generality by [10]

$$\boldsymbol{D}(\lambda) \equiv \tfrac{1}{2}[D^+(\lambda)D^{[1+N/2]-}(\lambda) + D^{[1+N/2]+}(\lambda)D^-(\lambda)]. \qquad (122)$$

In particular, for $V(q) = q^N + vq^{\frac{N}{2}-1}$ with $N \equiv 0\,[\mathrm{mod}4]$, by equation (100),

$$\boldsymbol{D}_N(0; v) \equiv \frac{1}{2}[D_N^+(0; v)D_N^-(0; -v) + D_N^+(0; -v)D_N^-(0; v)] \equiv \frac{\cos\pi\nu v}{\sin\pi\nu} \qquad (123)$$

(the explicit end result uses equation (120)). Thus, the values of $w = -v$ such that the potential $q^N - wq^{\frac{N}{2}-1}$ on the *whole real line* has a zero eigenvalue are also *exactly* quantized: they are the zeros of $\cos\frac{\pi}{N+2}v$, i.e.

$$w'_n = (N+2)(n+1/2) \quad (n \in \mathbb{Z}) \qquad \text{if} \quad N \equiv 0\ [\mathrm{mod}\ 4]. \qquad (124)$$

This is a spectrum naturally invariant under reflection; whereas in the even potential case $N \equiv 2$ [mod 4], the spectrum $\{w_k\}$ of equation (107) is recovered.

Curiously, the eigenfunctions (121) corresponding to the explicit generalized eigenvalues (124) do not seem to reduce to elementary functions, while they do so for the generalized eigenvalues (107); still, the basis that they form might prove useful (e.g. the $N = 4$ basis for general quartic anharmonic oscillators).

In conclusion, a unified analytical formalism has displayed the quantization of the special zero energy in potentials of the form $q^N + vq^{\frac{N}{2}-1}$ (beginning with $N = 4$), as well as the connection formula for the corresponding confluent hypergeometric functions, to be clear generalizations of the explicit exact quantization scheme for the harmonic oscillator. In contrast to the previous example, however, this analysis treats the problem entirely by known functions and does not generate any new ones.

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We are grateful to A Turbiner for suggesting we apply exact WKB analysis to quasi-exactly solvable potentials.

Appendix. Formulae for the potentials q^N

We recapitulate specific results and formulae, otherwise scattered in [2,3,5,7,11,24–29], about the spectral functions of the homogeneous Schrödinger operators on the half-line,

$$\hat{H}_N \stackrel{\text{def}}{=} -\frac{\mathrm{d}^2}{\mathrm{d}q^2} + q^N \qquad q \in [0, +\infty) \qquad N \geqslant 1 \text{ integer}. \tag{125}$$

(Not all N-dependences will be systematically stated.) As in section 1.2, we call $\hat{H}_N^+$ (respectively $\hat{H}_N^-$) the operator with the Neumann (respectively Dirichlet) condition at $q = 0$.

Three cases of special interest to us will be, on one hand, the harmonic ($N = 2$) and linear ($N = 1$) cases, both of which are describable using known special functions, as opposed to the nonelementary quartic case ($N = 4$). At the same time, $N = 4$ and 1 are both regular cases, and dual to each other (they share the same number of conjugates $L = 3$), whereas $N = 2$ stands out as a singular (confluent) case (also self-dual) [29].

A.1. General N

Notations:

$$D_N^{\pm}(\lambda) \stackrel{\text{def}}{=} \det(\hat{H}_N^{\pm} + \lambda) \qquad \text{and} \qquad \mu_N = \frac{N+2}{2N} \qquad \varphi = \frac{4\pi}{N+2}. \tag{126}$$

The set of exponents in equation (2) reduces to $\{0\} \cup \{(2n-1)\mu\}_{n=0,1,2,\ldots}$; the coefficient of the leading singularity $t^{-\mu}$ is

$$c_{-\mu}^{\pm} = (2\sqrt{\pi})^{-1}\Gamma(1 + 1/N). \tag{127}$$

In accordance with equation (5), the trace identities then have the pattern [3, 5]

$$Z_N(0) = 0 \qquad Z_N^{\mathrm{P}}(0) = \tfrac{1}{2} \tag{128}$$

$$\begin{cases} Z_N(-m) = 0 \qquad \text{unless} \quad m = \left(\dfrac{1}{2} + r\right)\left(1 + \dfrac{N}{2}\right) \\ Z_N^{\mathrm{P}}(-m) = 0 \qquad \text{unless} \quad m = r\left(1 + \dfrac{N}{2}\right) \end{cases} \qquad \text{for} \quad m \text{ and } r \in \mathbb{N}. \tag{129}$$

(Resulting asymptotic forms (8a) for $\log D_N^{\pm}(\lambda)$: cf [29, equation (11)].)

- As all the conjugate potentials $V^{[\ell]}(q)$ (equation (48)) coincide here, the main functional relation (50) for the spectral determinants boils down to [2]

$$e^{+i\varphi/4} D_N^+(e^{-i\varphi}\lambda) D_N^-(\lambda) - e^{-i\varphi/4} D_N^+(\lambda) D_N^-(e^{-i\varphi}\lambda) \equiv 2i \qquad (N \neq 2) \tag{130}$$

or, equivalently, to a multiplicative 'coboundary identity' linking the full and skew determinants, [11, 26, 28]

$$\begin{aligned} &D_N^P(\lambda)/D_N^P(e^{-i\varphi}\lambda) \equiv e^{i(-2\Phi_N(\lambda)+\varphi/2)} \\ &\Phi_N(\lambda) \stackrel{\text{def}}{=} \arcsin([D_N(e^{-i\varphi}\lambda) D_N(\lambda)]^{-1/2}) \end{aligned} \tag{131}$$

the branch of the arcsin being fixed at $\lambda = 0$ with the help of equation (130):

$$D_N(0) = (\sin\varphi/4)^{-1} \qquad \Longrightarrow \qquad \Phi_N(0) \stackrel{\text{def}}{=} \varphi/4. \tag{132}$$

- The exact quantization condition, drawn from equation (130), is [27–29]

$$\frac{2}{\pi} \arg D_N^{\pm}(-e^{-i\varphi}E)_{E=E_k} = k + \frac{1}{2} \pm \frac{N-2}{2(N+2)} \qquad \text{for} \quad k = {}^{0,2,4,\ldots}_{1,3,5,\ldots} \quad (N \neq 2). \tag{133}$$

- By multiplying together all the conjugates of equation (131), a corresponding multiplicative 'cocycle identity' (of length L given by equation (49)) results, providing a consistency condition upon D alone in the implicit form [26]

$$\sum_{\ell=0}^{L-1} \Phi_N(e^{-i\ell\varphi}\lambda) \equiv L\varphi/4 \tag{134}$$

i.e. an autonomous functional equation, circularly symmetric of order L (and convertible to a polynomial form) for the complete determinant D_N.

- Some special values of spectral functions also become explicit ([3, 11, appendix C]) thanks to the solvability of the differential equation $(\hat{H}_N + \lambda)\psi_\lambda(q) = 0$ at $\lambda = 0$, specifically here by Bessel functions [20, vol 2 Chapter 7.2.8]: the canonical recessive solution (39), of asymptotic behaviour $\psi_0(q) \sim q^{-N/4} \exp[-q^{1+N/2}/(1+N/2)]$ prescribed by equation (40), is

$$\psi_0(q) \equiv 2\sqrt{\nu/\pi}\, q^{1/2} K_\nu(2\nu q^{1+N/2}) \qquad \left(\nu \stackrel{\text{def}}{=} \frac{1}{N+2}\right) \tag{135}$$

(normalized by reference to the known $z \to +\infty$ behaviour of $K_\nu(z)$). In turn, the values $\psi_0(0)$, $\psi_0'(0)$ follow from $K_\nu(z) = \pi(2\sin\nu\pi)^{-1}[I_{-\nu}(z) - I_\nu(z)]$ plus $I_{\pm\nu}(z) \sim (z/2)^{\pm\nu}/\Gamma(\pm\nu+1)$ for $z \to 0$, and finally equation (47) yields

$$\begin{aligned} &D_N^+(0) = \frac{\Gamma(1-\nu)}{\nu^{N\nu/2}\sqrt{\pi}} \qquad D_N^-(0) = \frac{\Gamma(\nu)\nu^{N\nu/2}}{\sqrt{\pi}} \\ &\Longrightarrow \qquad D_N(0) = \frac{1}{\sin\nu\pi} \end{aligned} \tag{136}$$

(the last result agreeing with equation (132)).
Equation (135) also generates formulae for $Z_N^{\pm}(n)$, $n = 1, 2, \ldots$, by fully specifying a general expression of the 1D Green function,

$$\langle q|(\hat{H}+\lambda)^{-1}|q'\rangle = W_\lambda^{-1}\, \psi_\lambda(-\min\{q,q'\})\psi_\lambda(\max\{q,q'\}) \tag{137}$$

$$(W_\lambda \stackrel{\text{def}}{=} \text{Wronskian}\{\psi_\lambda(-q), \psi_\lambda(q)\}) \tag{138}$$

when $\hat{H} = \hat{H}_N$ and $\lambda = 0$ (with $W_\lambda \equiv 2D(\lambda)$ by equation (45), implying $W_0 = 2(\sin\varphi/4)^{-1}$ by equation (136)). Integral formulae giving $Z_N(n)$ as Tr $(\hat{H}_N^{-1})^n$, and

$Z_N^{\mathrm{P}}(n)$ as $\mathrm{Tr}\,\hat{P}(\hat{H}_N^{-1})^n$ ($\hat{P} \stackrel{\text{def}}{=}$ the parity operator), thus become explicit. For $n = 1$, the latter is the simpler:

$$Z_N^{\mathrm{P}}(1) = \frac{4\nu}{\pi} \sin \nu\pi \int_0^\infty [K_\nu(2\nu q^{1+N/2})]^2 q \, \mathrm{d}q \qquad \nu = \frac{1}{N+2}; \qquad (139)$$

finally this evaluates by a Weber–Schafheitlin formula [20, vol 2, chapter 7.14, equation (36)], and likewise for $Z_N(1)$ [3]:

$$Z_N^{\mathrm{P}}(1) = \frac{\sin \nu\pi}{2\sqrt{\pi}} (2\nu)^{2-4\nu} \frac{\Gamma(\nu)\Gamma(2\nu)\Gamma(3\nu)}{\Gamma(2\nu + \frac{1}{2})} \qquad Z_N(1) = \frac{\tan 2\nu\pi}{\tan \nu\pi} Z_N^{\mathrm{P}}(1). \qquad (140)$$

Remark. The results (136), (140) also work for $N = 1, 2$; —this approach can still handle $Z_N^{\pm}(2)$, but with final results reducible only to ${}_4F_3$ generalized hypergeometric series: [11, appendix C].

- For general integer n, by contrast, the farthest explicit algebraic result we can reach is a single identity (a sum rule) at the level of each doublet $(Z_N^{+}(n), Z_N^{-}(n))$, and this comes simply by expanding the functional identity (130) in all powers λ^n, as [11]

$$\sin\left[\frac{\varphi}{4} + \sum_{n=1}^{\infty} \sin\frac{n\varphi}{2} \frac{Z_N^{\mathrm{P}}(n)}{n}(-\lambda)^n\right] \equiv \exp\left[Z_N'(0) + \sum_{m=1}^{\infty} \cos\frac{m\varphi}{2} \frac{Z_N(m)}{m}(-\lambda)^m\right] \quad (141)$$

then equating both sides of this generating identity at each order n: at $n = 0$ we already obtained $D_N(0) = 1/\sin(\varphi/4)$ (equation (132) or (136)); then, at higher orders, we obtain sum rules in the form

$$\begin{aligned}
&\cot\frac{\varphi}{4} \sin\frac{\varphi}{2} Z_N^{\mathrm{P}}(1) - \cos\frac{\varphi}{2} Z_N(1) = 0 \qquad (N \neq 2) \\
&\cot\frac{\varphi}{4} \sin\frac{2\varphi}{2} Z_N^{\mathrm{P}}(2) - \cos\frac{2\varphi}{2} Z_N(2) = \left[2\cos\frac{\varphi}{4} Z_N^{\mathrm{P}}(1)\right]^2 \\
&\cot\frac{\varphi}{4} \sin\frac{3\varphi}{2} Z_N^{\mathrm{P}}(3) - \cos\frac{3\varphi}{2} Z_N(3) \\
&\qquad = 4\cos\frac{\varphi}{2}\left[3\cos\frac{\varphi}{2} Z_N^{\mathrm{P}}(1) Z_N^{\mathrm{P}}(2) - 2\cos^2\frac{\varphi}{4} Z_N^{\mathrm{P}}(1)^3\right] \\
&\vdots \\
&\cot\frac{\varphi}{4} \sin\frac{n\varphi}{2} Z_N^{\mathrm{P}}(n) - \cos\frac{n\varphi}{2} Z_N(n) = \text{a polynomial of } \{Z_N^{\pm}(m)\}_{1 \leqslant m < n}
\end{aligned} \qquad (142)$$

(the first line also follows from equation (140)).

Other results are resurgence properties of the spectral determinants (namely, exact analytical constraints on their $(1/\hbar)$-Borel transforms) [2, 11, 30].

A.2. Special cases

Out of the preceding results valid for general N, we only restate those which are specifically needed in the main text, or which take a particular form; special and numerical values are in table 1. We will be using $\mathrm{j} \stackrel{\text{def}}{=} \mathrm{e}^{2\mathrm{i}\pi/3}$.

A.2.1. $N = 4$. This case (quartic oscillator) has order $\mu = \frac{3}{4}$.
Main functional relation (130):

$$\mathrm{e}^{+\mathrm{i}\pi/6} D_4^+(\mathrm{j}^{-1}\lambda) D_4^-(\lambda) - \mathrm{e}^{-\mathrm{i}\pi/6} D_4^+(\lambda) D_4^-(\mathrm{j}^{-1}\lambda) \equiv 2\mathrm{i} \qquad (\varphi = 2\pi/3). \quad (143)$$

Exact quantization condition (133):

$$\frac{2}{\pi} \arg D_4^{\pm}(-\mathrm{j}^2 E)_{E=E_k} = k + \frac{1}{2} \pm \frac{1}{6} \qquad \text{for} \quad k = {}^{0,2,4,\ldots}_{1,3,5,\ldots}. \quad (144)$$

Cocycle functional equation (134): its algebraic form is

$$D_4(\lambda) D_4(\mathrm{j}\lambda) D_4(\mathrm{j}^2\lambda) \equiv D_4(\lambda) + D_4(\mathrm{j}\lambda) + D_4(\mathrm{j}^2\lambda) + 2. \quad (145)$$

Special values for $D_4^{\pm}(0)$, $Z_4^{\pm}(1)$: see table 1 (note that $D_4(0) = 2$).
The sum rules (142) can also be written for $N = 4$ as

$$\begin{aligned} &Z_4^+(1) - 2Z_4^-(1) = 0 \\ &2Z_4^+(2) - Z_4^-(2) = 3[Z_4^+(1) - Z_4^-(1)]^2 \\ &Z_4(3) = Z_4(1)^3/6 - Z_4(1)Z_4(2)/2 \qquad \text{etc.} \end{aligned} \quad (146)$$

(In addition, $Z_4^{\pm}(s)$ have been asymptotically evaluated for $s \to -\infty$, by exploiting the resurgence equations for the spectral determinants [2, 3].)

A.2.2. $N = 1$. The eigenvalues E_k are the (unsigned) zeros of the Airy functions, here meant as Ai (for odd parity) and Ai′ (for even parity) [21, Chapter 10.4]. A few results nevertheless seem new [29]. The order is $\mu = \frac{3}{2}$.

The canonical recessive solution simply relates to the Airy function:

$$\psi_\lambda(q) \equiv 2\sqrt{\pi}\,\mathrm{Ai}(q + \lambda). \quad (147)$$

The spectral determinants are then also Airy functions by equation (47),

$$\begin{aligned} &D_1^+(\lambda) = -2\sqrt{\pi}\,\mathrm{Ai}'(\lambda) \qquad D_1^-(\lambda) = 2\sqrt{\pi}\,\mathrm{Ai}(\lambda) \\ &\Longrightarrow \qquad D_1(\lambda) = -2\pi(\mathrm{Ai}^2)'(\lambda) \end{aligned} \quad (148)$$

and their functional relation (130) boils down to the classic Wronskian identity for $\mathrm{Ai}(\cdot)$ and $\mathrm{Ai}(\mathrm{j}^2\cdot)$:

$$\mathrm{e}^{+\mathrm{i}\pi/3} D_1^+(\mathrm{j}\lambda) D_1^-(\lambda) - \mathrm{e}^{-\mathrm{i}\pi/3} D_1^+(\lambda) D_1^-(\mathrm{j}\lambda) \equiv 2\mathrm{i} \qquad (\varphi = 4\pi/3). \quad (149)$$

Exact quantization condition (133):

$$\frac{2}{\pi} \arg D_1^{\pm}(-\mathrm{j}E)_{E=E_k} = k + \frac{1}{2} \mp \frac{1}{6} \qquad \text{for} \quad k = {}^{0,2,4,\ldots}_{1,3,5,\ldots}. \quad (150)$$

Cocycle identity: the algebraic form of (134) for $N = 1$ is

$$D_1(\lambda)^2 + D_1(\mathrm{j}\lambda)^2 + D_1(\mathrm{j}^2\lambda)^2 - 2[D_1(\mathrm{j}\lambda)D_1(\mathrm{j}^2\lambda) + D_1(\mathrm{j}^2\lambda)D_1(\lambda) + D_1(\lambda)D_1(\mathrm{j}\lambda)] + 4 \equiv 0. \quad (151)$$

Equations (136) for $D_1^{\pm}(0)$ just express the known values of Ai(0), Ai′(0) (see table 1).
Sum rules (142):

$$Z_1^+(1) = 0 \qquad Z_1^-(2) = Z_1^-(1)^2 \qquad Z_1(3) = 5Z_1(1)^3/2 - 3Z_1(1)Z_1(2)/2 \quad \text{etc.} \quad (152)$$

Here, due to a supplementary set of relations, all values $Z_1^{\pm}(n)$ become recursively expressible as rational functions of

$$\tau \stackrel{\text{def}}{=} D_1^{\mathrm{P}}(0) \equiv -\mathrm{Ai}'(0)/\mathrm{Ai}(0) = 3^{1/3}\Gamma(\tfrac{2}{3})/\Gamma(\tfrac{1}{3}) \approx 0.729\,011\,133; \quad (153)$$

they are all irrational except $Z_1^+(1) = 0$ (a regularized value, however) and $Z_1^+(3) = 1$ (table 1 lists all $Z_1^{\pm}(n)$ analytically up to $n = 3$).

A.2.3. $N = 2$. *The* finite-N case where the spectrum is known ($E_k = 2k+1$), of order $\mu = 1$ (an integer, which makes it a singular case).

The canonical recessive solution relates to a parabolic cylinder function [21, Chapter 19]; [20, vol 2 Chapter 8]

$$\psi_\lambda(q) \equiv 2^{(1-\lambda)/4} U(\lambda/2, \sqrt{2}q). \tag{154}$$

The determinants are given by gamma functions,

$$D_2^+(\lambda) = \frac{2^{-\lambda/2} 2\sqrt{\pi}}{\Gamma(\frac{1+\lambda}{4})} \qquad D_2^-(\lambda) = \frac{2^{-\lambda/2}\sqrt{\pi}}{\Gamma(\frac{3+\lambda}{4})}$$
$$\Longrightarrow \qquad D_2(\lambda) = \frac{2^{-\lambda/2}\sqrt{2\pi}}{\Gamma(\frac{1+\lambda}{2})} \tag{155}$$

their functional relation is now drawn directly from equation (50) instead of (130):

$$e^{+i\pi/4} D_2^+(-\lambda) D_2^-(\lambda) - e^{-i\pi/4} D_2^+(\lambda) D_2^-(-\lambda) \equiv 2ie^{i\pi\lambda/4} \qquad (\varphi = \pi) \tag{156}$$

and its right-hand side carries a special extra factor because for homogeneous potentials,

$$\beta_{-1}(s) \equiv 0 \qquad \text{except:} \qquad \beta_{-1}(s) \equiv \lambda(-s + \tfrac{1}{2}) \qquad \text{for} \quad N = 2. \tag{157}$$

For this special value $\varphi = \pi$, the functional relation further splits into its real and imaginary parts at real λ, and thereby reduces to

$$D_2^+(\lambda) D_2^-(-\lambda) \equiv 2\cos\frac{\pi}{4}(\lambda - 1) \tag{158}$$

hence it just amounts to the reflection formula for $\Gamma(z)$ (like the cocycle identity, which we do not write).

Here, the exact quantization condition $E_k = 2k + 1$ trivially arises by dispatching the obvious zeros of the right-hand side of equation (158): the positive ones to $D_2^-(-\lambda)$, the negative ones to $D_2^+(\lambda)$.

The anomalous value (157) accounts for special $N = 2$ trace identities:

$$Z_2^\pm(0, \lambda) \equiv (-\lambda \pm 1)/4 \tag{159}$$

the anomaly vanishes at (and only at) $\lambda = 0$, and indeed the general formulae (47), (136) for $D_N^\pm(\lambda = 0)$ correctly agree with equation (155) to give

$$D_2^+(0) = \frac{2\sqrt{\pi}}{\Gamma(\frac{1}{4})} \approx 0.977\,741\,067 \qquad D_2^-(0) = \frac{\sqrt{\pi}}{\Gamma(\frac{3}{4})} \approx 1.446\,409\,085$$
$$\Longrightarrow \qquad D_2(0) = \sqrt{2}. \tag{160}$$

The spectral zeta-functions are

$$Z_2(s) \equiv (1 - 2^{-s})\zeta(s) \qquad Z_2^{\mathrm{P}}(s) \equiv \beta(s)\left(\stackrel{\text{def}}{=} \sum_{k=0}^{\infty} (-1)^k/(2k+1)^s\right). \tag{161}$$

Their sum rules only reproduce the known special values $\zeta(2n)$ and $\beta(2n+1)$ [21, chapter 23],

$$Z_2^{\mathrm{P}}(1) = \pi/4 \qquad Z_2(2) = \pi^2/8 \qquad Z_2^{\mathrm{P}}(3) = \pi^3/32 \qquad \text{etc.} \tag{162}$$

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(Misprints in this paper: in equations (18), $D^{\pm}(\mathrm{e}^{-\mathrm{i}\varphi}\lambda)$ should read $D^{\pm}(-\mathrm{e}^{-\mathrm{i}\varphi}\lambda)$ (twice) and just afterwards, $[0, \mathrm{e}^{-\mathrm{i}\varphi}\infty)$ should read $[0, -\mathrm{e}^{-\mathrm{i}\varphi}\infty)$.)

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